Estimates for Singular Radon Transforms and Pseudodifferential Operators with Singular Symbols

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INTRODUCTION

There has been considerable interest in recent years in extending the theory of Calderón–Zygmund singular integrals to operators whose kernels are concentrated on (or singular along) submanifolds. Aside from the extensive work on Hilbert transforms along curves, L^2 and L^p estimates have been proven for translation-invariant operators on nilpotent Lie groups whose convolution kernels are singular both at the group identity element and along a submanifold of dimension $\geqslant 2$ by Geller and Stein [G-S], Müller [Mu I, Mu II], Greenleaf [G], and Ricci and Stein [R-S], where in the last three references the submanifold is not necessarily smooth at the identity. Nontranslation-invariant operators associated with a smoothly varying family of submanifolds have been introduced and studied by Phong and Stein [P-S I, P-S II]; they, however, assume that the submanifolds are smooth and nondegenerate in the sense that the conormal bundle of the singular support of the Schwartz kernel is locally the graph of a canonical transformation. This nondegeneracy condition they term "rotational curvature." Uhlmann [U] proved L^2 estimates for a class of pseudodifferential operators with singular symbols associated with two

cleanly intersecting Lagrangians, using the symbolic calculus for such class of operators which gives, as a corollary, the L^2 estimates of Phong and Stein.

The purpose of this paper is to simplify and unify the proofs of [P-S I, P-S II, U, G], and extend the results to certain nontranslation-invariant operators violating the smoothness and rotational curvature assumptions, examples of which arise naturally in integral geometry. We emphasize estimates on L^2 (or Sobolev spaces), but in fact the techniques handle easily the analytically continued operators which are used to prove L^p boundedness.

In Section 1 we consider various spaces of distributions defined in terms of iterated regularity which are sufficient to deal with the operators of Phong and Stein and examine their representations as oscillatory integrals with product type symbols. In order to handle operators associated with cones and powers of real principal type operators, we also review the spaces of Fourier integral distributions associated with two cleanly inter- secting Lagrangians, introduced by Melrose and Uhlmann [M-U] and Guillemin and Uhlmann [Gu-U], and the composition calculus (under certain geometric assumptions) of Antoniano and Uhlmann [A-U]. The proof of the L^2 boundedness of the Phong–Stein operators, presented in Section 2, depends on the crucial observation of Melrose, in unpublished lecture notes [M I] which have influenced our treatment considerably, that such an operator can be decomposed, via a parabolic microlocal cutoff, into the sum of a pseudodifferential operator with symbol (or, more accurately, amplitude) of type $(\frac{1}{2}, \frac{1}{2})$ and a Fourier integral operator with amplitude of type $(\frac{1}{2}, \frac{1}{2})$. The L^2 boundedness of the pseudodifferential operator follows immediately from the Calderón–Vaillancourt theorem; on the other hand, it can be shown that the composition of a Fourier integral operator, associated with a canonical graph and with amplitude of type $(\frac{1}{2}, \frac{1}{2})$, with its adjoint is a pseudodifferential operator of type $(\frac{1}{2}, \frac{1}{2})$, thereby yielding the L^2 boundedness of the Fourier integral operator. This result on compositions is essentially already in Beals [B], but not quite in the form we need, since we are interested in Fourier integral operators given by general phase functions and not just a generating function of the canonical transformation. For the sake of completeness we provide a proof, which follows closely the usual analysis of the composition of Fourier integral operators, the main novelty being that Hörmander's theorem on the invariance of classes of oscillatory integrals hold for type $(\frac{1}{2}, \frac{1}{2})$.

It should be noted that the decomposition of a singular Radon transform into two pieces, $T = T_1 + T_2$, with T_1 and $T_2 T_2^*$ both "classical" singular integral operators and hence bounded on L^2, already occurs in the early work of Nagel, Stein, and Wainger [N-S-W] on Hilbert transforms along variable curves in the plane.

Singular Radon transforms on a manifold X belong to the class of Fourier integral operators associated with two cleanly intersecting Lagrangians, $I^{p,l}(\Delta, \Lambda)$, with Δ being the diagonal in $T^*X \times T^*X$ and $\Lambda = N^* Z'$, where Z is the support of the Schwartz kernel. (Elements of $I^{p,l}(\Delta, \Lambda)$ are sometimes referred to as pseudodifferential operators with singular symbols, since microlocally on $\Delta \backslash \Lambda$ they are pseudodifferential operators whose principal symbols are singular at $\Delta \cap \Lambda$.) One can weaken the smoothness assumption on Z at the diagonal of $X \times X$ and still have Λ be a smooth Lagrangian; this happens, for example, for variable families of cones satisfying a curvature condition. The decomposition argument above is not available if Λ is not a canonical graph, since $\Lambda' \circ \Lambda$ is not the diagonal (and may not even be smooth). However, there is a situation of maximum degeneracy, namely when Λ is the flowout of an involutive (coisotropic) submanifold of $T^*X \backslash 0$, for which there is a composition calculus for $I^{p,l}(\Delta, \Lambda)$, due to Antoniano and Uhlmann [A-U]. Using this calculus, we establish in Section 3 the L^2 boundedness of elements of $I^{p,l}(\Delta, \Lambda)$, Λ a flowout, and then formulate classes of singular Radon transforms associated with variable families of cones to which this applies. The flowout condition is automatic in the translation-invariant case; it should be pointed out that the "geometric" proof of the boundedness in [G] is really a special case of the argument here (in disguise). The results here are in some sense complementory to those of [R-S, Mu I, Mu II] mentioned above, since for a translation-invariant operator on a non-commutative nilpotent group, Λ is usually not a flowout.

Finally, in Section 4 we apply the results of Section 3, in combination with those of Greenleaf and Uhlmann [G-U], to obtain some estimates in integral geometry. If (M, g) is an n-dimensional riemannian manifold, we may (at least locally) form the $(2n - 2)$-dimensional manifold $\mathcal{M}$ of geodesics on M and define the X-ray transform

$$\mathcal{R}f(\gamma) = \int_{\mathbb{R}} f(\gamma(s)) \, ds, \qquad \gamma \in \mathcal{M}, \, f \in C_0^\infty(M).$$

An n-dimensional submanifold $\mathscr{C} \subset \mathcal{M}$ is called a geodesic complex. Following Gelfand, one can form the restricted X-ray transform $\mathcal{R}_\mathscr{C} f = \mathcal{R}f|_\mathscr{C}$ and ask to what extent $\mathcal{R}_\mathscr{C} f$ determines f. In [G-U], it was shown that if $\mathscr{C}$ satisfies a generalization of Gelfand's admissibility criterion, $\mathcal{R}_\mathscr{C}$ has a relative left parametrix constructed from a relative parametrix for $\mathcal{R}_\mathscr{C}^t \circ \mathcal{R}_\mathscr{C}$. In fact, subject to a curvature hypothesis, $\mathcal{R}_\mathscr{C}^t \circ \mathcal{R}_\mathscr{C} \in I^{p,l}(\Delta, \Lambda)$, with Gelfand's criterion implying that Λ is a flowout. The results of Section 3 can then be used to derive Sobolev space estimates for $\mathcal{R}_\mathscr{C}$; in particular, there is a loss of $\frac{1}{4}$ derivative, reflecting the particular way that the conormal bundle of the point-geodesic relation fails to be a

canonical graph. (There are closely related results in Guillemin [Gu II], where, moreover, Λ need not be smooth.) On the other hand, for "many" complexes in general position, which do not satisfy Gelfand's criterion, $\mathcal{R}_{\mathscr{C}}$ satisfies a *better* estimate: there is a loss of only $\frac{1}{6}$ derivative, which follows from results of Melrose and Taylor [M-T] on folding canonical relations.

Phong and Stein have informed us that they have reproven their estimates, using a parabolic cutoff that seems to be different from ours.[1] L^p estimates for some restricted X-ray transforms in $\mathbb{R}^n$ are in Wang [W].

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1. OPERATOR CLASSES

In this section we shall review the pertinent facts from the theory of product-type conormal distributions and the associated distributions that will be used in this paper. First we recall the definition of a classical conormal distribution [H II, p. 4].

DEFINITION 1.1. Let X be a C^∞ manifold of dimension n, and $S \subset X$ a smooth submanifold. The space of conormal distributions on X with respect to S of order m, denoted $I^m(X; S)$, is the set of all distributions $u \in \mathscr{D}'(X)$ such that

$$V_1 \cdots V_k u \in H_{\mathrm{loc}}^{-m-n/4, \infty}(X), \qquad \forall k \geqslant 0, \tag{1.2}$$

where the V_j's are C^∞ vector fields on X which are tangent to S, and $H_{\mathrm{loc}}^{s,\infty}(X)$ is the usual Besov space. Since we will not be using this definition and its generalizations to find the exact order of distributions, we will work with Sobolev spaces rather than Besov spaces.

When we have two submanifolds (or subvarieties) $S_1, S_2 \subseteq X$, we can define a space of conormal distributions associated with $S_1 \cup S_2$ allowing interaction at $S_1 \cap S_2$, under the assumption that the conormal bundles $N^* S_1, N^* S_2$ are smooth and intersect cleanly in $T^* X$. This space was defined using oscillatory integrals with singular symbols in [M-U, Gu-U]; we shall review this approach below, but first we wish to consider the case where S_1 and S_2 are smooth and intersect cleanly. We shall follow here the notes of Melrose [M-U] and define this space of product-type conormal distributions using iterated regularity, in analogy with (1.1). Recall

DEFINITION 1.3. $S_1, S_2 \subset X$ intersect cleanly if $S_1 \cap S_2$ is smooth and $T(S_1 \cap S_2) = TS_1 \cap TS_2$.

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[1] This has appeared. See [P-S III].

For s mplicity, we shall restrict our attention to nested submanifolds

$$S_2 \subseteq S_1 \subseteq X, \tag{1.4}$$

which i; relevant to the Phong–Stein operators. Of course, in this case S_1 and S_2 intersect cleanly.

DEFINITION 1.5. Let $\mathscr{V}(S_1, S_2)$ be the space of smooth vector fields on X which are tangent to both S_1 and S_2.

(Note that clean intersection is exactly the right condition to make "tangent to both S_1 and S_2" unambiguous.)

DEFINITION 1.6. The space of product-type conormal distributions with respect to S_1, S_2, denoted by $I(X; S_1, S_2)$, is the set of all $u \in \mathscr{D}'(X)$ such that

$$V_1 \cdots V_k u \in H^{s_0}_{\text{loc}}(X) \qquad \text{for some } s_0 \in \mathbb{R} \text{ and } \forall k \geqslant 0 \tag{1.7}$$

with $V_j \equiv \mathscr{V}(S_1, S_2)$, $1 \leqslant j \leqslant k$.

It is straightforward to prove

PROPOSITION 1.8. If $u \in I(X; S_1, S_2)$, then $WF(u) \subseteq N^*S_1 \cup N^*S_2$. Moreover, away from S_2, $u \in I(X; S_1 \backslash S_2)$.

We will assume from now on that S_1 is of codimension d_1 and S_2 is of codimension $d_1 + d_2$. Introduce local coordinates near a point of S_2 such that

$$\begin{cases} S_1 = \{x_1 = \cdots = x_{d_1} = 0\} \\ S_2 = \{x_1 = \cdots = x_{d_1 + d_2} = 0\}, \end{cases} \tag{1.9}$$

and denote points in $\mathbb{R}^n$ by (x', x'', x'''), with $x' = (x_1, ..., x_{d_1})$, $x'' = (x_{d_1 + 1}, ..., x_{d_1 + d_2})$, $x''' = (x_{d_1 + d_2 + 1}, ..., x_n)$; let (ξ', ξ'', ξ''') be the dual variables. We now find a local basis for $\mathscr{V}(S_1, S_2)$, and thus for the ring of differential operators generated by $\mathscr{V}(S_1, S_2)$.

PROPOSITION 1.10. If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $S_2 \subseteq S_1 \subseteq \mathbb{R}^n$ are as in (1.9), then $u \in I(\mathbb{R}^n; S_1, S_2)$ iff there exists an $s_0 \in \mathbb{R}$ such that

$$D^\alpha_{x'} D^\beta_{x''} D^\gamma_{x'''} ((x')^\rho (x'')^\delta u) \in H^{s_0}_{\text{loc}}(\mathbb{R}^n),$$

$\forall$ multiindices $\alpha, \beta, \gamma, \delta, \rho$ such that $|\rho| \geqslant |\alpha|$, $|\delta| + |\rho| \geqslant |\alpha| + |\beta|$. (1.11)

Proof. It is enough to show that the differential operators in (1.11) form a local basis for the ring generated by $\mathscr{V}(S_1, S_2)$. This will be proven

by induction on the order of the operator. First, we claim that the vector fields

$$D_{x_k} \quad \text{for} \quad k > d_1 + d_2; \qquad x_i D_{x_j} \quad \text{for} \quad i \leqslant d_1 \text{ if } j \leqslant d_1$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (1.12)

$$i \leqslant d_1 + d_2 \quad \text{if} \quad d_1 < j \leqslant d_1 + d_2$$

give a basis for $\mathscr{V}(S_1, S_2)$ (over $C^\infty(\mathbb{R}^n)$). To see this, note that if $V = \sum_{i=1}^{d_1} a_i(x) D_{x_i} + \sum_{j=d_1+1}^{d_1+d_2} b_j(x) D_{x_j} + \sum_{k=d_1+d_2+1}^{n} c_k(x) D_{x_k}$, with $a_i, b_j, c_k \in C^\infty(\mathbb{R}^n)$, then V tangent to S_1 implies that the a_i's must vanish at $x' = 0$, while V tangent to S_2 means that the a_i's and b_j's must vanish at $(x', x'') = 0$.

Now, suppose by induction that for some l, the operators in (1.11) with $|\alpha| + |\beta| + |\gamma| \leqslant l$ span the $C^\infty(\mathbb{R}^n)$-submodule of the ring generated by $\mathscr{V}(S_1, S_2)$ consisting of those operators of order $\leqslant l$. If $V_j \in \mathscr{V}(S_1, S_2)$, $1 \leqslant j \leqslant l + 1$, then

$$V_1 \cdots V_{l+1} = \sum a_{\alpha\beta\gamma}^{\rho\delta ij} D_{x'}^\alpha D_{x''}^\beta D_{x'''}^\gamma ((x')^\rho (x'')^\delta \phi_{ij} x_i D_{x_j})$$

$$+ \sum b_{\alpha\beta\gamma}^{\rho\delta k} D_{x'}^\alpha D_{x''}^\beta D_{x'''}^\gamma ((x')^\rho (x'')^\delta \phi_k D_{x_k}), \qquad (1.13)$$

where the sum extends over $|\alpha| + |\beta| + |\gamma| \leqslant l$, with the other indices limited as in (1.11) and (1.12), and the a's and ϕ's are C^∞ functions. Commuting the D_{x_j}, $j \leqslant d_1$, and D_{x_k}, $k > d_1 + d_2$, past the x', x'', ϕ factors then gives a sum of terms as in (1.11), with $|\alpha| + |\beta| + |\gamma| \leqslant l + 1$. ∎

The above can now be used to give an alternate characterization of $I(\mathbb{R}^n; S_1, S_2)$, with S_1 and S_2 as in (1.9), in terms of oscillatory integrals with symbol-valued symbols.

DEFINITION 1.14. Let (ξ', ξ'') be coordinates on $\mathbb{R}^{d_1^*} \times \mathbb{R}^{d_2^*}$. For $m \in \mathbb{Z}^+$, $M, M' \in \mathbb{R}$, the space of symbol-values symbols of order M, M' on $\mathbb{R}^m \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, denoted by $S^{M,M'}(\mathbb{R}^m; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$, is the space of smooth functions $a(x, \xi', \xi'')$ satisfying, for every compact $K \subset \mathbb{R}^m$,

$$\sup_{x \in K} |D_{\xi'}^\alpha D_{\xi''}^\beta D_x^\gamma a(x, \xi', \xi'')| \leqslant C_{k,\alpha,\beta,\gamma} \langle \xi', \xi'' \rangle^{M - |\alpha|} \langle \xi'' \rangle^{M' - |\beta|},$$

$$\forall \text{ multiindices } \alpha, \beta. \qquad (1.15)$$

Here, $\langle \xi'_1, \xi'' \rangle = (1 + |\xi'_1|^2 + |\xi''|^2)^{1/2}$, $\langle \xi'' \rangle = (1 + |\xi''|^2)^{1/2}$.

Using standard integration by parts arguments (cf. [H1]), to each $a \in S^{M,M'}(\mathbb{R}^m; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ we can, if $m \geqslant d_1 + d_2$, associate an $I_a \in \mathscr{D}'(\mathbb{R}^m)$ defined by the oscillatory integral

$$I_a = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} e^{i(x' \cdot \xi' + x'' \cdot \xi'')} a(x, \xi', \xi'') \, d\xi' \, d\xi''. \qquad (1.16)$$

These give us the second characterization of $I(\mathbb{R}^n; S_1, S_2)$.

PROPOSITION 1.17.   *Let* $u \in I(\mathbb{R}^n; S_1, S_2)$, *with* S_1 *and* S_2 *as in* (1.9). *Then* $u = I_a$ *with* $a \in S^{M, M'}(\mathbb{R}^n; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ *for some* M, M'.

Proof   We can assume without loss of generality that $u \in \mathscr{E}'(\mathbb{R}^n) \cap I(\mathbb{R}^n; S_1, S_2)$. Taking the partial Fourier transform of u in the x', x'' variables and using (1.11), we obtain that

$$(\xi')^\alpha (\xi'')^\beta D_{x'''}^\gamma D_{\xi'}^\rho D_{\xi''}^\delta \hat{u}(\xi', \xi'', x''')$$

$$\in L^2(\mathbb{R}^{d_1^*} \times \mathbb{R}^{d_2^*} \times \mathbb{R}^{n - d_1 - d_2}; \langle \xi', \xi'' \rangle^{s_0} d\xi' d\xi'' dx''') \qquad (1.18)$$

for some $s_0 \in \mathbb{R}$, when $|\rho| \geqslant |\alpha|$, $|\delta| + |\rho| \geqslant |\alpha| + |\beta|$. By the Sobolev embedding theorem, we get that $a(x, \xi', \xi'') = \hat{u}(\xi', \xi'', x''')$ satisfies (1.15) for some $M, M' \in \mathbb{R}$. ∎

DEFINITION 1.19.   For $M, M' \in \mathbb{R}$, S_1 and S_2 as in (1.9), $I^{M, M'}(\mathbb{R}^n; S_1, S_2)$ is the space of oscillatory integrals of the form (1.16) where S_1, S_2 are given locally by (1.9) with $a \in S^{M, M'}(\mathbb{R}^n; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$.

We next show that this definition is actually coordinate free.

PROPOSITION 1.20.   *The space* $I^{M, M'}(\mathbb{R}^n; S_1, S_2)$, *with* S_1, S_2 *as in* (1.4), *is independent* (*modulo* $C^\infty(\mathbb{R}^n)$) *of the choice of local coordinates*.

Proof   Suppose that u is of the form (1.16) with the integral absolutely convergent. A change of local coordinates that preserves (1.9) necessarily has

$$x_i = \sum_{j=1}^{d_1} A_{ij}(y) y_j, \qquad 1 \leqslant i \leqslant d_1$$

$$x_i = \sum_{j=1}^{d_1 + d_2} B_{ij}(y) y_j, \qquad d_1 < i \leqslant d_1 + d_2$$

with A_{ij}, B_{ij} smooth. Inserting this in (1.16) and introducing

$$\Xi_i = \sum_{j=1}^{d_1} A_{ji}\xi_j + \sum_{j=d_1+1}^{d_1+d_2} B_{ji}\xi_j, \qquad 1 \leqslant i \leqslant d_1,$$

$$\Xi_i = \sum_{j=d_1+1}^{d_1+d_2} B_{ji}\xi_j, \qquad d_1 < i \leqslant d_1 + d_2,$$

one obtains the new representation

$$u(y) = \int e^{i(\Xi' \cdot y' + \Xi'' \cdot y'')} b(y, \Xi', \Xi'') \, d\Xi' \, d\Xi'', \qquad (1.21)$$

with $b(y, \Xi', \Xi'') = a(x(y), \xi'(y, \Xi', \Xi''), \xi''(y, \Xi', \Xi''))$ times the Jacobian of the change of variable $(\xi', \xi'') \to (\Xi', \Xi'')$. It is easy to see that $b \in S^{M, M'}(\mathbb{R}^n; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$. For general values of M, M', for which the integral is divergent, one as usual integrates by parts to reduce the calculation to the convergent case. ∎

DEFINITION 1.22. Let $S_2 \subseteq S_1 \subseteq X$ be nested manifolds with $\dim X = n$ and S_1, S_2 of codimensions $d_1, d_1 + d_2$ in X, respectively. Then $I^{M, M'}(X; S_1, S_2)$ is the space of locally finite sums of distributions of the form (1.16) where S_1, S_2 are given locally by (1.9).

Now let X be as above and let $\Lambda \subset T^* X \backslash 0$ be a conic Lagrangian manifold. Let $I^m(X; \Lambda)$ be the space of Fourier integral distributions on X associated with Λ [H I]; an element of $I^m(X; \Lambda)$ is given as a locally finite sum of expressions

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x, \theta)} a(x, \theta) \, d\theta, \tag{1.23}$$

where the phase function ϕ parameterizes Λ and a is a symbol of order $m - N/2 + n/4$. By slight abuse of notation, we will say that a $u \in \mathscr{D}'(X)$ is in $I^m(X; \Lambda)$ if for each $\lambda_0 \in \Lambda$ there is a microlocalization of u near λ_0 which belongs to $I^m(X; \Lambda)$. When we wish to emphasize the symbol class of which $a(x, \theta)$ in (1.23) belongs, we will write $I^m_{\rho, \delta}(X; \Lambda)$; otherwise, $\rho = 1$, $\delta = 0$ is understood.

PROPOSITION 1.24. Let $u \in I^{M, M'}(X; S_1, S_2)$ as in (1.19). Then $u \in I^{M + M' + (d_1 + d_2)/2 - n/4}_{1, 0}(X; N^* S_2 \backslash N^* S_1)$ and $u \in I^{M + d_1/2 - n/4}_{1, 0}(X; N^* S_1 \backslash N^* S_2)$.

Proof. In local coordinates with S_1 and S_2 as in (1.9), we have the representation (1.16) with $a \in S^{M, M'}(\mathbb{R}^n; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$. In these coordinates, $N^* S_1 = \{(0, x'', x'''; \xi', 0, 0): x'' \in \mathbb{R}^{d_2}, x''' \in \mathbb{R}^{n - d_1 - d_2}$ and $\xi' \in \mathbb{R}^{d_1^*}\}$ and $N^* S_2 = \{(0, 0, x'''; \xi', \xi'', 0): x''' \in \mathbb{R}^{n - d_1 - d_2}, \xi' \in \mathbb{R}^{d_1^*}$ and $\xi'' \in \mathbb{R}^{d_2^*}\}$. Let $\Sigma = N^* S_1 \cap N^* S_2$; then $\Sigma = \{(0, 0, x'''; \xi', 0, 0): x''' \in \mathbb{R}^{n - d_1 - d_2}, \xi' \in \mathbb{R}^{d_1^*}\}$. On $N^* S_1 \backslash \Sigma$, we have $x'' \neq 0$ and repeated integration by parts in the ξ'' variable shows that we can lower the order of a arbitrarily in the ξ'' variable and so obtain (modulo C^∞) the microlocal representation

$$u = \int e^{ix' \cdot \xi'} b(x, \xi') \, d\xi', \qquad b \in S^M_{1, 0}(\mathbb{R}^n \times \mathbb{R}^{d_1}). \tag{1.25}$$

Thus, $u \in I^{M + d_1/2 - n/4}_{1, 0}(X; N^* S_1 \backslash N^* S_2)$. On the other hand, on $N^* S_2 \backslash \Sigma$, we have $\xi'' \neq 0$ and so $a \in S^{M + M'}_{1, 0}(\mathbb{R}^n \times \mathbb{R}^{d_1 + d_2})$ there by (1.15), yielding $u \in I^{M + M' + (d_1 + d_2)/2 - n/4}_{1, 0}(X; N^* S_2 \backslash N^* S_1)$. ∎

For more general classes of product-type conormal distributions, the iterated regularity definition using vector fields in (1.6) is not applicable. For example, in treating singular integral operators with conical singularities in Section 3, we will deal with the case that S_1 has a conical singularity; it may happen that $\mathcal{V}(S_1, S_2)$ is empty. Also, more generally, we may associate classes of distributions to intersecting Lagrangians which are not necessarily conormal bundles. We now describe the spaces of distributions associated with pairs of Lagrangians, defined in [M-U, Gu-U], for a particular case, namely when one of the Lagrangians is the diagonal in $T^*X \times T^*X$. These are sometimes referred to as pseudo-differential operators with singular symbols.

Let X be of dimension n and $\Delta \subset (T^*X \backslash 0) \times (T^*X \backslash 0)$ the diagonal; Δ' is Lagrangian for the product symplectic form $\pi_1^* \omega_{T^*X} + \pi_2^* \omega_{T^*X}$, and Δ' contains the wave front set of a pseudodifferential operator on X. Let $\Lambda \subset (T^*X \backslash 0) \times (T^*X \backslash 0)$ be another conic Lagrangian. We assume that

$$\Delta \text{ and } \Lambda \text{ intersect cleanly.} \tag{1.26}$$

Let $\Sigma = \Delta \cap \Lambda$ and denote the codimension of Σ in Δ (and Λ) by k, $1 \leqslant k \leqslant 2n - 1$. Consider a model case where $X = \mathbb{R}^n$, $\tilde{\Delta} = \{(x, \xi; x, \xi)\}$ and

$$\tilde{\Lambda} = \{(x, \xi; y, \eta): x'' = y'', \xi' = \eta' = 0, \xi'' = \eta''\}. \tag{1.27}$$

Here we are denoting a point of $\mathbb{R}^n$ by $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then $\Sigma = \{x' = y', x'' = y'', \xi' = \eta' = 0, \xi'' = \eta''\}$.

DEFINITION 1.28. For $m \in \mathbb{Z}^+$ and $p, l \in \mathbb{R}$, the space of product-type symbols denoted $\tilde{S}^{p,l}(\mathbb{R}^m; \mathbb{R}^n, \mathbb{R}^k)$ is the set of all smooth functions on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k$ such that for all $K \subset \mathbb{R}^m$ compact,

$$|D_\xi^\alpha D_\sigma^\beta D_x^\gamma a(x, \xi, \sigma)| \leqslant C_{\alpha\beta\gamma}(1 + |\xi|)^{p - |\alpha|}(1 + |\sigma|)^{l - |\beta|}. \tag{1.29}$$

DEFINITION 1.30. The class of operators $I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda})$ consists of those mappings $A: \mathscr{E}(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ with Schwartz kernels

$$K_A(x, y) = \int e^{i((x' - y' - s) \cdot \xi' + (x'' - y'') \cdot \xi'' + s \cdot \sigma)} a(x, y, s, \xi, \sigma) \, d\sigma \, ds \, d\xi, \tag{1.31}$$

with $a \in \tilde{S}^{p - n/2 + k/2, l - k/2}(\mathbb{R}^{2n+k}; \mathbb{R}^n, \mathbb{R}^k)$.

We will identify elements of $I^{p,l}$ with their Schwartz kernels without comment. For $A \in I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda})$, it is straightforward to see that the analogues of (1.8) and (1.24) hold (see [Gu-U]):

$$WF(A)' \supseteq \tilde{\Delta} \cup \tilde{\Lambda} \quad \text{and} \quad A \in I^{p+l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta} \backslash \Sigma), A \in I^p(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Lambda} \backslash \Sigma).$$

From the proof, one obtains the microlocal representation of A on $\tilde{\Delta}\backslash\Sigma$ (modulo terms of lower order)

$$K_A(x, y) = \int e^{i(x-y)\cdot\xi} a(x, y, 0, \xi, \xi')\, d\xi$$

which, given the product nature of a, exhibits A as a pseudodifferential operator with singular symbol. The following results, proved in [Gu-U], allow the definition of $I^{p,l}(X \times X; \Delta, \Lambda)$ in general.

PROPOSITION 1.32.   *Let* $\tilde{\chi}: T^*(\mathbb{R}^n \times \mathbb{R}^n)\backslash 0 \to T^*(\mathbb{R}^n \times \mathbb{R}^n)\backslash 0$ *be a cano-nical transformation such that* $\tilde{\chi}(\tilde{\Delta}) \subset \tilde{\Delta}$, $\tilde{\chi}(\tilde{\Lambda}) \subset \tilde{\Lambda}$, *and let* F *be a Fourier integral operator of order* 0 *associated with* $\tilde{\chi}$. *Then* $F(I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda})) \subset I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda})$.

PROPOSITION 1.33.   *Given* $\Delta, \Lambda \subset (T^*X\backslash 0) \times (T^*X\backslash 0)$ *intersecting cleanly in codimension* k, *there exists a canonical transformation* χ: $(T^*X\backslash 0) \times (T^*X\backslash 0) \to (T^*\mathbb{R}^n\backslash 0) \times (T^*\mathbb{R}^n\backslash 0)$ *such that* $\chi(\Delta) \subseteq \tilde{\Delta}$, $\chi(\Lambda) \subseteq \tilde{\Lambda}$ *(for the same* k*)*.

Thus, one defines $I^{p,l}(X \times X; \Delta, \Lambda)$ to be those operators whose Schwartz kernels are locally finite sums of $F(K_A)$'s, $A \in I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda})$ and F associated with χ as in (1.33).

We can now give an iterated regularity characterization of $I^{p,l}(X \times X; \Delta, \Lambda)$ as in (1.6), but now using first order pseudodifferential operators instead of vector fields. Note first that by representing the Schwartz kernel $u = K_A(x, y)$ of an $A \in I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda})$ as in (1.31), we may obtain an oscillatory integral representation of u as an element of $I^{\cdot,\cdot}(\mathbb{R}^n \times \mathbb{R}^n; S_1, S_2)$, where $S_1 = \{(x', x'', y', y''): x'' = y''\}$ and $S_2 = \Delta_{\mathbb{R}^n} = \{(x', x'', y', y''): x' = y', x'' = y''\}$. To avoid confusing notation, introduce coordinates $z' = x'' - y'' \in \mathbb{R}^{n-k}$, $z'' = x' - y' \in \mathbb{R}^k$, and $z''' = x + y \in \mathbb{R}^n$ on $\mathbb{R}^n \times \mathbb{R}^n$. Thus, $d_1 = n - k$, $d_2 = k$. We have

$$u(z) = \int e^{i(z'\cdot\xi' + z''\cdot\xi'')}\left(\int e^{is\cdot(\sigma - \xi'')} a(x, y, s, (\xi'', \xi'), \sigma)\, ds\, d\sigma\right) d\xi'\, d\xi''.$$

$$(1.34)$$

The inner integral may be evaluated using stationary phase and the symbol estimates (1.29); the result, $b(x, y, \xi', \xi'')$, lies in $S^{M,M'}(\mathbb{R}^n \times \mathbb{R}^n; S_1, S_2)$ for $M = p - n/2 + k/2$, $M' = l - k/2$. Thus, $I^{p,l}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{\Delta}, \tilde{\Lambda}) = I^{p-n/2+k/2, l-k/2}(\mathbb{R}^n \times \mathbb{R}^n; S_1, S_2)$. Since the latter space is characterized by iterated regularity via vector fields tangent to S_1 and S_2 as in (1.6), so is the former. Now, if F is a Fourier integral operator of order zero associated with a χ

as in (1.33), and $V \in \mathscr{V}(S_1, S_2)$, then $V \circ F^{-1}$ is a first-order pseudo-differential operator P on $X \times X$ which is characteristic for $\Delta \cup \Lambda$, i.e., $\sigma_{\mathrm{prin}}(P) = 0$ on $\Delta \cup \Lambda$. We thus are lead to

PROPOSITION 1.35.   *If $u \in \mathscr{D}'(X \times X)$ and there exists an $s_0 \in \mathbb{R}$ such that*

$$P_1 \cdots P_k u \in H_{\mathrm{loc}}^{s_0}(X \times X), \qquad all \quad k \geqslant 0, \qquad (1.36)$$

for all first order P_j with $\sigma_{\mathrm{prin}}(P_j) = 0$ on $\Delta \cup \Lambda$, then $u \in I^{p,l}(X \times X; \Delta, \Lambda)$ for some $p, l \in \mathbb{R}$.

For the results of Sections 3 and 4, we shall also need the symbol calculus for $I^{p,l}(X \times X; \Delta, \Lambda)$, and a composition calculus assuming that Λ satisfies a certain geometric condition. We will describe these briefly; the reader is referred to [M-U, Gu-U] for more details.

If $u \in I^{p,l}(X \times X; \Delta, \Lambda)$, by the above discussion $u \in I^{p+l}(X \times X; \Delta \backslash \Sigma)$ and $u \in I^p(X \times X; \Lambda \backslash \Sigma)$; thus, by the standard theory of Fourier integral operators, u has invariantly defined symbols $\sigma_0(u)$ and $\sigma_1(u)$ on $\Delta \backslash \Sigma$ and $\Lambda \backslash \Sigma$, respectively. Because of the product-type estimates satisfied by the amplitude used to define u, $\sigma_0(u)$ has a singularity at Σ, and in fact $\sigma_0(u)$ is conormal for Σ on Δ, belonging to the space of sections $R^{l-k/2}(\Omega_0 \otimes L_0; \Delta, \Sigma)$ singular of order $l - k/2$ defined in [Gu-U, p. 260]. The space $S^{p,l}(X \times X; \Delta, \Sigma)$ is defined to be those elements of $R^{l-k/2}$ which are homogeneous of degree $p + l + n/2$. If $u \in I^{p,l}(X \times X; \Delta, \Lambda)$, then $\sigma_0(u) \in S^{p,l}(X \times X; \Delta, \Sigma)$, and the symbol calculus is summarized by

PROPOSITION 1.37.   *The following sequence is exact:*

$$0 \to I^{p,l-1}(X \times X; \Delta, \Lambda) + I^{p-1,l}(X \times X; \Delta, \Lambda)$$

$$\to I^{p,l}(X \times X; \Delta, \Lambda) \xrightarrow{\ \sigma_0\ } S^{p,l}(X \times X; \Delta, \Sigma) \to 0.$$

If $\Lambda \subset (T^*X \backslash 0) \times (T^*X \backslash 0)$ is a canonical relation, i.e., is a conic Lagrangian for the difference symplectic form $\pi_1^* \omega_{T^*X} - \pi_2^* \omega_{T^*X}$, we denote the class of operators with Schwartz kernels in $I^{p,l}(X \times X; \Delta, \Lambda')$ by $I^{p,l}(\Delta, \Lambda)$. From [Gu-U, Proposition 6.2] we have that $\bigcap_l I^{p,l}(\Delta, \Lambda) = I^p(\Lambda)$, the space of classical Fourier integral operators of order p associated with Λ, and $\bigcap_p I^{p,l}(\Delta, \Lambda) = C_0^\infty(X \times X)$, the space of smoothing operators. In order to obtain a composition calculus for $I^{p,l}(\Delta, \Lambda)$ we need to restrict Λ so that new wave front set does not occur in the composition. Recall that a submanifold $\Gamma \subset T^*X \backslash 0$ is involutive (or coisotropic) if $\Gamma = \{(x, \xi) : p_i(x, \xi) = 0, \ 1 \leqslant i \leqslant k\}$, with the p_i's defining functions for Γ that are in involution at Γ: all the Poisson brackets $\{p_i, p_j\}$ vanish on Γ.

Γ is then foliated by the integrable distribution consisting of the span of the H_{p_i}'s. The flowout of Γ is

$$\Lambda_\Gamma = \left\{ (x, \xi; y, \eta) \in T^*X \times T^*X : (x, \xi) \in \Gamma, (y, \eta) \right.$$

$$= \exp\left(\sum_{j=1}^{k} t_j H_{p_j} \right)(x, \xi), \; t \in \mathbb{R}^k \Bigg\}. \tag{1.38}$$

Λ_Γ is a canonical relation (if Γ is conic). For Λ of the form Λ_Γ, the composition calculus is summarized by

PROPOSITION 1.39. *If $\Lambda = \Lambda_\Gamma$ for a conic, involutive $\Gamma \subset T^*X \backslash 0$ of codimension k, then $I^{p,l}(\Lambda, \Lambda) \circ I^{p',l'}(\Lambda, \Lambda) \subseteq I^{p+p'+k/2, l+l'-k/2}(\Lambda, \Lambda)$, and $\sigma_0(A \circ B) = \sigma_0(A) \cdot \sigma_0(B)$.*

2. THE CANONICAL GRAPH CASE

In this section we prove local L^2 estimates for operators whose Schwartz kernels lie in the class $I^{M,M'}(X \times X; S_1, S_2)$, where S_2 is the diagonal of a smooth n-dimensional manifold X, and $S_1 \supseteq S_2$ is smooth and such that N^*S_1' is a (local) canonical graph. We then show that the singular Radon transforms of Phong and Stein [P-S I, P-S II] belong to this class, as well as the operators arising in the analytic-interpolation proof of L^p estimates. The L^2 estimates for these operators were previously reproven by Uhlmann [U] using the symbolic calculus (1.37) developed in [Gu-U]. Here, we give a simpler proof, making use of an observation of Melrose [M I] that elements of $I(X \times X; S_1, S_2)$ can be decomposed into a sum of two classical Fourier integral distributions, conormal for S_1 and S_2, respectively, but with amplitudes of type $(\frac{1}{2}, \frac{1}{2})$. The idea of a parabolic cutoff goes back to Boutet de Monvel [Bo]; in this context, it was used by Guillemin [Gu I] in defining singular symbols.

PROPOSITION 2.1 (Melrose). *Let $S_2 \subseteq S_1 \subseteq X \times X$, with $\dim X = n$, codim $S_1 = d_1$, and $\operatorname{codim} S_2 = d_1 + d_2$. Then, if $-d_2 < M' \leqslant 0$,*

$$I^{M,M'}(X \times X; S_1, S_2) \subseteq I^{M''}_{1/2,1/2}(X \times X; S_1) + I^{M''}_{1/2,1/2}(X \times X; S_2),$$

where $M'' = M + M'/2 + (1/2)(d_1 + d_2 - n)$.

Proof. Recall that for $S \subset X \times X$, $I^m_{\rho,\delta}(X \times X; S)$ is just different notation for Hörmander's class $I^m_{\rho,\delta}(X \times X; N^*S)$. In local coordinates, $u \in I^{M,M'}(X \times X; S_1, S_2)$ can be represented as in (1.16):

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x' \cdot \xi' + x'' \cdot \xi'')} a(x, \xi', \xi'') \, d\xi' \, d\xi'',$$

with $a \in S^{M,M'}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^{d_1}, \mathbb{R}^{d_2})$, where (x', x'', x''') are coordinates on $X \times X$ such that $S_1 = \{x' = 0\}$, $S_2 = \{x' = x'' = 0\}$.

Pick a cutoff function $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \equiv 1$ near 0, and set

$$u_1 = \int e^{i(x' \cdot \xi' + x'' \cdot \xi'')} \varphi\left(\frac{\langle \xi'' \rangle^2}{\langle \xi', \xi'' \rangle}\right) a(x, \xi', \xi'')\, d\xi'\, d\xi'' \qquad (2.2)$$

and $u_2 = u - u_1$. On the support of $(1 - \varphi)(\langle \xi'' \rangle^2 / \langle \xi', \xi'' \rangle)$, we have $\langle \xi'' \rangle \geqslant c\langle \xi', \xi'' \rangle^{1/2}$ and the symbol estimate (1.15) becomes

$$|\partial_\xi^\alpha \partial_{\xi''}^\beta \partial_x^\gamma ((1 - \varphi) a)| \leqslant c\langle \xi', \xi'' \rangle^{M + M'/2 - |\alpha| - |\beta|/2}, \qquad (2.3)$$

which is of type $(\frac{1}{2}, 0)$. Of course, this condition is not coordinate invariant and so we get that u_2 has a representation as a conormal distribution for S_2 with an amplitude of order $M + M'/2$ and type $(\frac{1}{2}, \frac{1}{2})$. Thus, $u_2 \in I_{1/2,1/2}^{M + M'/2 + (d_1 + d_2)/2 - n/2}(X \times X; S_2)$.

On the other hand, write u_1 as a conormal distribution for S_1 with amplitude

$$b(x, \xi') = \int_{\mathbb{R}^{d_1}} e^{ix'' \cdot \xi''} \varphi\left(\frac{\langle \xi'' \rangle^2}{\langle \xi', \xi'' \rangle}\right) a(x, \xi', \xi'')\, d\xi''. \qquad (2.4)$$

Then $\partial_\xi^\alpha \partial_x^\gamma b(x, \xi')$ will be a sum of terms, the leading one of which will be

$$\int_{\mathbb{R}^{d_2}} e^{ix'' \cdot \xi''} (i\xi'')^\gamma \varphi\left(\frac{\langle \xi'' \rangle^2}{\langle \xi', \xi'' \rangle}\right) \partial_\xi^\alpha a(x, \xi', \xi'')\, d\xi''. \qquad (2.5)$$

Noting that the integral is over a ball of radius $\leqslant c\langle \xi' \rangle^{1/2}$ in $\mathbb{R}^{d_2}$ and simply estimating the integrand by its absolute value via (1.15), we may dominate the integral by $c\langle \xi' \rangle^{M - |\alpha'|} \int_1^{\langle \xi' \rangle^{1/2}} r^{M' + |\gamma''| + d_2 - 1}\, dr \leqslant c\langle \xi' \rangle^{M - M'/2 + d_2/2 - |\alpha'| + |\gamma''|/2}$ if $M' + d_2 > 0$. The other terms in $\partial_\xi^\alpha \partial_x^\gamma b(x, \xi')$ are handled similarly; thus, u_1 has a representation as a conormal distribution for S_1 with amplitude of order $M + M'/2 + d_2/2$ and type $(\frac{1}{2}, \frac{1}{2})$, and so $u \in I_{1/2,1/2}^{M + M'/2 + d_1/2 + d_2/2 - n/2}(X \times X; S_1)$. ∎

When S_2 is the diagonal Δ of X in $X \times X$, we have written an element of $I^{M,M'}(X \times X; S_1, \Delta)$ as a sum of a pseudodifferential operator of type $(\frac{1}{2}, \frac{1}{2})$ and a Fourier integral operator associated with the Lagrangian $N^* S_1$ with amplitude of type $(\frac{1}{2}, \frac{1}{2})$. The Calderón–Vaillancourt theorem will allow us to prove the main result of this section.

THEOREM 2.6. *Let $\Delta \subseteq S_1 \subseteq X$ with $\dim X = n$, $\operatorname{codim} S_1 = d$. Suppose $N^* S_1' \subset (T^* X \backslash 0) \times (T^* X \backslash 0)$ and is a local canonical graph. Then, if $A \in I^{M,M}(X \times X; S_1, \Delta)$, then*

$$A : L^2_{\mathrm{comp}}(X) \to L^2_{\mathrm{loc}}(X)$$

if $\max(M', M + M'/2) \leqslant 0$ and $M' > d - n$.

Proof. Here, $d_1 = d$, $d_2 = n - d$, and $d_1 + d_2 = n$. As noted above, by (2.1), A can be written as a sum of a Fourier integral operator A_1 of order $M + M'/2$ and type $(\frac{1}{2}, \frac{1}{2})$, and a pseudodifferential operator of order $M + M'/2$ and type $(\frac{1}{2}, \frac{1}{2})$. The Calderón–Vaillancourt theorem [C-V] applies directly to the latter to give L^2 boundedness. The boundedness of A_1 is reduced to the Calderón–Vaillancourt theorem by the result that the composition of a Fourier integral operator, associated with a canonical graph and having amplitude of type $(\frac{1}{2}, \frac{1}{2})$, with its adjoint is a pseudo-differential operator of type $(\frac{1}{2}, \frac{1}{2})$. This is a special case of a theorem of Beals [B, Theorem 5.4] for general weights, but the proof in [B] is actually only for "canonical operators," where the phase function is a generating function of the canonical transformation. Since we need the result for Fourier integral operators with general phase functions, as in (2.8), we present the (somewhat different) proof in full.

PROPOSITION 2.7. *Let $X \subset \mathbb{R}^n$ be open and A_1 be a properly supported operator on X with Schwartz kernel*

$$K_{A_1}(x, y) = \int_{\mathbb{R}^m} e^{i\phi_1(x, y, \theta)} a_1(x, y, \theta) \, d\theta \qquad (2.8)$$

with $\phi_1(x, y, \theta)$ a nondegenerate phase function parameterizing a Lagrangian Λ_ϕ such that Λ'_ϕ is a canonical graph. Let $a_1 \in S_{1/2, 1/2}^{(n-m)/2}(X \times X; \mathbb{R}^m)$. Then, modulo smoothing operators, $A_1 A_1^*$ is a pseudodifferential operator of order 0 and type $\frac{1}{2}, \frac{1}{2}$.

Proof. We follow closely the treatment of composition of Fourier integral operators in Duistermaat [D, pp. 57–60]; this requires only checking the dependence of the proof on the type of the symbol.

First note that $A_2 = A_1^*$ is again a Fourier integral operator with amplitude of type $\frac{1}{2}, \frac{1}{2}$ with representation

$$A_2 f(x) = \int e^{-i\phi(y, x, \theta)} \bar{a}_1(y, x, \theta) f(y) \, d\theta \, dy. \qquad (2.9)$$

Let $a_2(x, y, \theta) = \bar{a}_1(y, x, \theta)$ and $\phi_2(x, y, \theta) = -\phi_1(y, x, \theta)$; ϕ_2 parameterizes $\Lambda_{\phi_1}^{-1}$. As in [D], we write the Schwartz kernel of $A_1 \circ A_1^* = A_1 \circ A_2$ as an oscillatory integral

$$K_{A_1 \circ A_2}(x, z) = \int e^{i(\phi_1(x, y, \theta) + \phi_2(y, z, \sigma))} a_1(x, y, \theta) a_2(y, z, \sigma) \, d\theta \, d\sigma \, dy. \qquad (2.10)$$

We introduce cutoff functions χ_1, χ_2, supported where $|\sigma| \leqslant c |\theta|$, $|\theta| \leqslant c |\sigma|$,

respectively; on their supports we have $|d_{(y,\theta)}(\phi_1 + \phi_2)| \geqslant c'|\theta|$, $|d_{(y,\sigma)}(\phi_1 + \phi_2)| \geqslant c'|\sigma|$, respectively. As in [D], the integrals

$$\int e^{i(\phi_1 + \phi_2)} \chi_j a_1 a_2 \, d\theta \, d\sigma \, dy, \qquad j = 1, 2,$$

are C^∞ functions of x and z: one first integrates in σ (resp. θ), and then uses the rapid oscillation of the exponential in the remaining variables to integrate by parts, which completely overwhelm the loss of $\frac{1}{2}$ when differentiating in x or z. We are thus left with the main term

$$\tilde{K}_{A_1 \circ A_2}(x, z) = \int e^{i(\phi_1(x, y, \theta) + \phi_2(y, z, \sigma))} b(x, y, z, \theta, \sigma) \, d\theta \, d\sigma \, dy, \quad (2.11)$$

with $b = (1 - \chi_1 - \chi_2) a_1 a_2$ being supported where $|\theta| \sim |\sigma|$. One introduces a new phase variable $\tilde{y} = |(\theta, \sigma)| \, y$, phase function $\phi(x, z, \theta, \sigma, \tilde{y}) = \phi_1(x, \tilde{y}/(\theta, \sigma)|, \theta) + \phi_2(\tilde{y}/|(\theta, \sigma)|, z, \sigma)$, and amplitude

$$a(x, z, \theta, \sigma, \tilde{y}) = |(\theta, \sigma)|^{-n} b\left(x, \frac{\tilde{y}}{|(\theta, \sigma)|}, z, \theta, \sigma\right);$$

one easily checks that $a \in S_{1/2, 1/2}^{-M}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^{2m+n})$. Forthermore, ϕ is a nondegenerate phase function that parameterizes the diagonal. Modulo C^∞, we have

$$K_{A_1 \circ A_1^*}(x, z) = \int_{\mathbb{R}^{2m+n}} e^{i\phi(x, z, \xi)} a(x, z, \xi) \, d\xi, \qquad (2.12)$$

where ϕ parameterizes the diagonal and a is of type $\frac{1}{2}, \frac{1}{2}$ and of the correct order to make $A_1 A_1^*$ of order 0. It remains to show that we can replace ϕ by the usual parameterization of the diagonal, $\phi_0(x, z, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and a by an $a_0 \in S_{1/2, 1/2}^0(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. This follows from the fact that Hörmander's result on the equivalence of classes of oscillatory integrals defined by different phase functions parameterizing the same Lagrangian still holds for amplitudes of type $(\frac{1}{2}, \frac{1}{2})$. Following Hörmander [H I, pp. 142–147], we first decrease the number of phase variables as much as possible. The key point is that in the integral (3.2.4) of [H I], stationary phase is replaced by integration by parts. That is, if $\theta \in \mathbb{R}^N$ is denoted by $(\theta', \theta'') \in \mathbb{R}^{N-K} \times \mathbb{R}^k$, $a(x, \theta) \in S_{1/2, 1/2}^M(\mathbb{R}^n \times \mathbb{R}^N)$ is supported in $\{|\theta''| \leqslant c|\theta'|\}$, and $Q(\theta'', \theta'')$ is a nondegenerate quadratic form on $\mathbb{R}^k$, then

$$b(x, \theta') = \int_{\mathbb{R}^k} e^{iQ(\theta'', \theta'')} a(x, \theta', \theta'') \, d\theta'' \qquad (2.13)$$

belongs to $S_{1/2, 1/2}^{M+k/2}(\mathbb{R}^n \times \mathbb{R}^{N-k})$. By a rotation in $\mathbb{R}^k$, we may assume that $Q(\theta'', \theta'') = \sum_{j=1}^k \lambda_j \theta_j^2$, with each $\lambda_j \neq 0$. Let $\chi \in C^\infty(\mathbb{R})$, with $\chi \equiv 0$ near 0

and $\chi \equiv 1$ near ∞, and set $a_1(x, \theta', \theta'') = (1 - \chi)(|\theta''|/|\theta'|^{1/2}) \, a(x, \theta', \theta'')$, $a_2(x, \theta', \theta'') = \chi(|\theta''|/|\theta'|^{1/2}) \, a(x, \theta', \theta'')$. With $b_1(x, \theta')$, $b_2(x, \theta')$ defined by (2.13), using a_1, a_2, respectively, we have $b = b_1 + b_2$. The first term we simply estimate by

$$|b_1(x, \theta')| \leqslant \int_{|\theta''| \leqslant c |\theta'|^{1/2}} |a(x, \theta', \theta'')| \, d\theta'' \leqslant c(1 + |\theta'|)^{m + k/2}.$$

For the second term, note that $a_2 \in S^m_{1/2, 1/2}(\mathbb{R}^n \times \mathbb{R}^N)$. Define a differential operator $L = |\theta''|^{-2} \sum_{j=1}^k (\theta_j/2i\lambda_j)(\partial/\partial\theta_j)$, so that $L(e^{iQ}) = e^{iQ}$; one has $L^t = L + c |\theta''|^{-2}$. If $\alpha \in S^\mu_{1/2, 1/2}(\mathbb{R}^n \times \mathbb{R}^N)$, then $L\alpha \in (1/|\theta''|) \, S^{\mu - 1/2}_{1/2, 1/2}$, while $(c/|\theta''|^2) \alpha \in (1/|\theta''|^2) \, S^\mu_{1/2, 1/2}$. Using L, integrate by parts M times in the expression for $b_2(x, \theta')$:

$$b_2(x, \theta') = \int_{c |\theta'|^{1/2} \leqslant |\theta''| \leqslant c' |\theta'|} e^{iQ(\theta'', \theta'')}(L^t)^M \, (a_2(x, \theta', \theta'')) \, d\theta''. \qquad (2.14a)$$

The integrand is dominated by a sum of terms of the form $|\theta''|^{j - 2M} (1 + |\theta'| + |\theta''|)^{m - 1/2}$, $0 \leqslant j \leqslant M$; introducing polar coordinates, the corresponding term of $b_2(x, \theta')$ is dominated by

$$\int_{c |\theta'|^{1/2}}^{c' |\theta'|} (1 + |\theta'| + r)^{m - j/2} \, r^{j - 2M + k} \, \frac{dr}{r}$$

$$\leqslant (1 + |\theta'|)^{m + k + 1/2 - 2M} \int_{c |\theta'|^{-1/2}}^{c'} (1 + r)^{m - j/2} \, r^{j - 2M + k} \, \frac{dr}{r}$$

$$\leqslant c(1 + |\theta'|)^{m + k/2 - M} \qquad \text{if} \quad j - 2M + k < 0.$$

Thus, choosing $M > k$ arbitrarily large, we find that $b_2(x, \theta')$ is rapidly decreasing, and so $|b(x, \theta')| \leqslant c(1 + |\theta'|)^{m + k/2}$. Derivatives of $b(x, \theta')$ are of the form (2.13), with a replaced by a derivative, and are handled in the same manner, yielding $b \in S^{M + k/2}_{1/2, 1/2}(\mathbb{R}^n \times \mathbb{R}^{N - k})$. In the application to (2.12) (where the spatial variables are denoted by (x, z)), the number of phase variables is decreased in this way to the minimum possible for a pseudo-differential operator, namely n, attained by the standard phase $\phi_0(x, z, \xi) = (x - z) \cdot \xi$. The resulting expression for the Schwartz kernel of $A_1 A_1^*$ is

$$K_{A_1 A_1^*}(x, z) = \int_{\mathbb{R}^n} e^{i\tilde\phi(x, z, \theta')} b(x, z, \theta') \, d\theta', \qquad (2.14b)$$

with $b \in S^0_{1/2, 1/2}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ and $\tilde\phi$ parameterizing the diagonal. By [H I, Theorem 3.16], $\tilde\phi$ and ϕ_0 are equivalent in the sense that there exists a diffeomorphism $\Phi \colon \mathbb{R}^{2n} \times (\mathbb{R}^n \backslash 0) \to \mathbb{R}^{2n} \times (\mathbb{R}^n \backslash 0)$, homogeneous of degree 1 in the last variable, so that $\tilde\phi \circ \Phi = \phi_0$. Making the corresponding change of

variable in (2.14b), we preserve the symbol class and obtain a representation of $A_1 A_1^*$ as a pseudodifferential operator of order 0 and type $(\frac{1}{2}, \frac{1}{2})$. This finishes the proof of Proposition 2.7. ∎

Proposition 2.7 immediately implies Theorem 2.6 when N^*S_1 is a canonical graph. When N^*S_1 is merely a local canonical graph, we form microlocal partitions of unity $\sum_j P^j(x, D) = I = \sum_k Q^k(y, D)$, such that on the support of each $p^j(x, \xi) \cdot q^k(y, \eta)$, N^*S_1 is a canonical graph. Then $A_1 = \sum_{j,k} P^j A_1 Q^k = \sum_{j,k} A_1^{jk}$; each A_1^{jk} is a Fourier integral operator with amplitude of type $\frac{1}{2}, \frac{1}{2}$ to which (2.7) applies, finishing the proof of (2.6).

We now relate the above results to the singular Radon transforms of Phong and Stein. We are going to follow closely their notation; in particular, for simplicity we limit ourselves to the case of hypersurfaces. A singular Radon transform, R, is defined by integrating a function defined on a smooth manifold X of dimension $n + 1$ along a hypersurface X_p passing through each point $p \in X$ against a distribution supported on X_p having a singularity of the type of a Calderón–Zygmund singular integral at p. A nondegeneracy condition, called rotational curvature, is imposed on the family of X_p's. Explicitly, using local coordinates (t, x, s, y) on $X \times X$, with $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, the hypersurface through a point $p = (t, x)$ is given by

$$X_p = \{(s, y): s = t + S(t, x, y), \ y \in \mathbb{R}^n\}, \tag{2.15}$$

where $S: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is smooth, $S(t, x, x) = 0$.

Let $S = \{(t, x, s, y) \in X \times X: t - s + S(t, x, y) = 0\}$, $S_2 = \Delta_X$. Then $S_2 \subseteq S_1 \subseteq X \times X$ with codim $S_1 = d_1 = 1$, codim $S_2 = d_2 + d_1 = n + 1$. "Rotational curvature" is

$$d^2_{x, y} S(t, x, x) \text{ is nondegenerate}, \qquad \forall x \in X. \tag{2.16}$$

This is coordinate invariant and, as noted in [P-S I], is equivalent with the condition that N^*S_1' be a local canonical graph near Δ_{T^*X}. The singular Radon transform is defined by

$$Rf(t, x) = \int_{\mathbb{R}^n} K(t, x, x - y) f(t + S(t, x, y), y) \, dy, \tag{2.17}$$

$K(t, x, \cdot)$ being a smooth family of Calderón–Zygmund kernels on $\mathbb{R}^n$. The Schwartz kernel of R is

$$K_R(t, x, s, y) = \delta(t - s + S(t, x, y)) K(t, x, x - y). \tag{2.18}$$

Since $\delta(\ )$ is a conormal distribution for the origin on $\mathbb{R}^1$, and a Calderón–Zygmund kernel is conormal for the origin in $\mathbb{R}^n$, it is natural to expect

that K_R belongs to a product-type conormal space as described in Section 1. In fact, choose coordinates $(z', z'', z''') \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n+1}$ on $X \times X$ such that $S_1 = \{z' = 0\}$ and $S_2 = (z' = z'' = 0\}$. Clearly, there is an $s_0 \in \mathbb{R}$ such that $K_R \in H_{\text{loc}}^{s_0}(X \times X)$ for all choices of K. To show that $K_R \in I(X \times X; S_1, S_2)$, we must show that all the iterates $V_1 \cdots V_k K_R \in H_{\text{loc}}^{s_0}$ $(X \times X)$ as well, where each $V_j \in \mathscr{V}(S_1, S_2)$. By (1.11) it is enough to show that $D_{z'}^\alpha D_{z''}^\beta D_{z'''}^\gamma ((z')^\rho (z'')^\delta K_R) \in H_{\text{loc}}^{s_0}(X \times X)$ for all $\alpha, \beta, \gamma, \delta, \rho$ such that $|\rho| \geqslant |\alpha|$, $|\delta| + |\rho| \geqslant |\alpha| + |\beta|$. But since K_R is a δ-function in z', the expression in question is zero if $|\rho| \geqslant 1$; thus we need only consider $\rho = 0$, and thus $\alpha = 0$. The question is then reduced to whether $D_{z''}^\beta D_{z'''}^\gamma$ $((z'')^\delta K_R) \in H_{\text{loc}}^{s_0}$ for all $|\delta| \geqslant |\beta|$. But this follows from (1.2) and the fact that K is conormal for the diagonal.

To compute M and M' for K_R, we use (1.24). Since R, away from the diagonal, is a generalized Radon transform associated to the family X_p of hypersurfaces [Gu-S], it is a Fourier integral operator of order $-(\dim X_p)/2 = -n/2$ on $N^* S_1 \backslash N^* S_2$. Hence $M - n/2 = -n/2$, so $M = 0$. Microlocally away from $N^* S_1$, R is easily seen to be a pseudodifferential operator of order 0, and thus $M' = 0$, putting $K_R \in I^{0,0}(X \times X; S_1, S_2)$. The L^2 boundedness of R then follows from (2.6).

To prove the boundedness of R on L^p, $1 < p < \infty$, Phong and Stein use an analytic family of operators, R_γ, $\gamma \in \mathbb{C}$, for which R_0 is essentially R. Let $\phi_\gamma(\cdot)$ be an analytic family of distributions on $\mathbb{R}$ with Fourier transform smooth near 0 and in the classical symbol class $S_{\text{cl}}^{\text{Re}(\gamma)}(\mathbb{R})$. They define

$$R_\gamma f(z, x) = \int K_\gamma(t, x, x - y) f(t + S(t, x, y)) \, dy \qquad (2.19)$$

with $K_\gamma(t, x, x - y) = |x - y|^{-2} \; \phi_\gamma(|x - y|^{-2} \; (t - s + S(t, x, y)))$ $K(t, x, x - y)$.

For $\text{Re}(\gamma) > 0$, R_γ is an operator of Calderón–Zygmund type on X, considered as a space of homogeneous type for a family of parabolic balls, and so is bounded on L^p, $1 < p < \infty$, while for $\text{Re}(\gamma) > 3/4 - n/2$, it is proved in [P-S II] that R_γ is bounded on L^2. Analytic interpolation is then employed to establish the L^p boundedness of R_0, and hence R.

We modify this analytic continuation slightly by first straightening out S_1 in $X \times X$. Let (z', z'', z''') be the coordinates on $X \times X$ introduced above, so that $S_1 = \{z' = 0\}$. Then $K_R(z) = \delta(z') \otimes K(z'', z'''; z')$, where K is conormal for $z'' = 0$ and thus has an oscillatory representation $K(z; z'') = \int_{\mathbb{R}^n} e^{iz'' \cdot \xi''} a(z; \xi'') \, d\xi''$ with $a \in S^0(X \times X; \mathbb{R}^n)$. (By using a smooth cutoff to restrict ourselves to a compact subset of $X \times X$, we can assume that a is smooth at $\xi'' = 0$.) This gives an explicit representation of K_R as an element of $I^{0,0}(X \times X; S_1, S_2)$:

$$K_R(z) = \int_{\mathbb{R}^{n+1}} e^{i(z' \cdot \zeta' + z'' \cdot \zeta'')} 1(\zeta') \cdot a(z; \xi'') \, d\zeta' \, d\xi'', \qquad (2.20)$$

with $1 \cdot a \in S^{0,0}(X \times X; \mathbb{R}^1, \mathbb{R}^n)$. Now define

$$\tilde{K}_\gamma(z) = |z''|^{-2} \phi_\gamma \left(\frac{z'}{|z''|^2} \right) K(z; z''), \qquad \gamma \in \mathbb{C}. \qquad (2.21)$$

First taking the partial Fourier transform in ξ', obtaining $\hat{\phi}_\gamma(|z''|^2 \xi')$ $K(z; z'')$, and then taking the Fourier transform in ξ'', making use of the classical type of $\hat{\phi}_\gamma$, we get that

$$\tilde{K}_\gamma(z) = \int_{\mathbb{R}^{n+1}} e^{i(z' \cdot \xi' + z'' \cdot \xi'')} a_\gamma(z; \xi', \xi'') \, d\xi' \, d\xi'', \qquad (2.22)$$

with $a_\gamma \equiv S^{-\operatorname{Re}(\gamma), 2\operatorname{Re}(\gamma)}(X \times X; \mathbb{R}, \mathbb{R}^n)$. Thus, by (2.6) we obtain the local L^2 boundedness of the operator $\tilde{R}_\gamma$ with Schwartz Kernel K_γ for $-n/2 < \operatorname{Re}(\gamma) < 0$, improving slightly the result of Phong and Stein. On the other hand, for $\operatorname{Re}(\gamma) > 0$, $\tilde{R}_\gamma$ is still a Calderón–Zygmund operator for a family of balls satisfying the Vitali covering condition, since a diffeomorphism of $X \times X$ fixing the diagonal pointwise does not change this condition for a family of *parabolic* balls (cf. Nagel and Stein [N-S]). We have thus recovered the result of Phong and Stein:

THEOREM 2.23. *Let R be as in* (2.17). *Then*

$$R: L^p_{\text{comp}}(X) \to L^p_{\text{loc}}(X), \qquad 1 < p < \infty.$$

3. THE FLOWOUT CASE AND OPERATORS WITH CONICAL SINGULARITIES

We now turn to proving boundedness of operators in the class $I^{p,l}(\Delta, \Lambda)$ when $\Lambda \subset (T^*\mathbb{R}^n\backslash 0) \times (T^*\mathbb{R}^n\backslash 0)$ is a flowout, using the composition calculus described in Section 1. Interesting examples of such operators will be furnished by singular integral operators associated with variable families of cones satisfying a certain tangency condition.

Let $\Sigma \subset (T^*\mathbb{R}^n\backslash 0, \omega)$ be a smooth, codimension k conic submanifold, $1 \leqslant k < n$, which is involutive:

> the ideal of smooth functions vanishing on Σ is closed under
> the Poisson bracket.   (3.1)

Thus, $T_{x,\xi}\Sigma^\omega \subset T_{(x,\xi)}\Sigma$ is a k-plane for all $(x, \xi) \in \Sigma$, and, as described in Section 2, the distribution $\{T_{(x,\xi)}\Sigma^\omega\}_{(x,\xi) \in \Sigma}$ is integrable, with integral

submanifolds $\Xi_{(x,\xi)}$, called the bicharacteristic leaves of Σ. The flowout of Σ is the canonical relation $\Lambda_\Sigma = \Lambda \subset (T^*\mathbb{R}^n\backslash 0) \times (T^*\mathbb{R}^n\backslash 0)$ given by

$$\Lambda_\Sigma = \{(x, \xi, y, \eta) \in \Sigma \times \Sigma : (y, \eta) \in \Xi_{(x,\xi)}\}. \tag{3.2}$$

Note that the projections $\pi_1, \pi_2 : \Lambda \to T^*\mathbb{R}^n\backslash 0$ have constant rank $2n - k$.

THEOREM 3.3. *Let $A \in I^{p,l}(\mathbb{R}^n, \mathbb{R}^n; \Lambda, \Lambda)$, with $\Lambda = \Lambda_\Sigma$ as in (3.2). Then $A : H^s_{\text{comp}}(\mathbb{R}^n) \to H^{s+s_0}_{\text{loc}}(\mathbb{R}^n)$ continuously, $\forall s \in \mathbb{R}$, if*

$$\max\left(p + \frac{k}{2}, p + l\right) \leqslant -s_0. \tag{3.4}$$

Proof. Since $I^{p'}(\Lambda) \circ I^{p,l}(\Lambda, \Lambda) \subset I^{p + p',l}(\Lambda, \Lambda)$, we may assume that $s_0 = 0$ and $\max(p + k/2, p + l) = 0$. We may further suppose that A is properly supported. By the composition calculus of (1.39), the product A^*A lies in $I^{p',l'}(\Lambda, \Lambda)$, with $p' = 2p + k/2$, $l' = 2l - k/2$ still satisfying $\max(p + k/2, p + l) = 0$. Furthermore, the principal symbol is $\sigma_0(A^*A) = |\sigma_0(A)|^2 \geqslant 0$. We now repeat in this setting a standard proof of the L^2 boundedness of pseudodifferential operators, due to Hörmander [H I]. (This method of proof was used in [U] for the operators considered in Section 2; it is also implicit in the geometrical proof for cones in [G].) Namely, we will construct a B such that

$$A^*A + B^*B = c^2 I \qquad \text{mod } I^{-1/2}(\Lambda), \tag{3.5}$$

for some $c > 0$. By the result of Hörmander on L^2 boundedness of Fourier integral operators associated with canonical relations that drop rank by at most k [H I, p. 182], an element of $I^{-k/2}(\Lambda)$ is bounded from $L^2_{\text{comp}}(\mathbb{R}^n)$ to $L^2_{\text{loc}}(\mathbb{R}^n)$, so that (4.4) implies the L^2 boundedness of A.

For a fixed compact $K \subset \mathbb{R}^n$, we will consider A acting on distributions supported in K. For $m \in \mathbb{R}$, set

$$I_m = \bigoplus_{\substack{p + l = m \\ p \leqslant -1/2}} I^{p,l}(\Lambda, \Lambda), \tag{3.6}$$

where the right hand side consists of finite sums, so that $m \geqslant m' \Rightarrow I_m \supseteq I_{m'}$ and $\bigcap_m I_m = I^{-k/2}(\Lambda)$. We now consider two cases. If $p + l = 0$, so that $l \geqslant k/2$, let c be any real number greater than $\limsup_{t \to \infty} |\sigma_0(A)(x, t\xi)|$ for all $(x, \xi) \in (T^*\mathbb{R}^n\backslash 0)|_K$, and let $b_0 = (c^2 - |\sigma_0(A)|^2)^{1/2}$. Then $b_0 \in S^{-k/2,k/2}$ $(\mathbb{R}^n \times \mathbb{R}^n; \Lambda, \Sigma)$ and by the symbol calculus (1.3) there is an operator $B_0 \in I^{-k/2,k/2}(\Lambda, \Lambda)$ with $\sigma_0(B_0) = b_0$. By (1.37) and (1.39),

$$\tilde{B}_1 = A^*A + B_0^*B_0 - c^2 I \in I_{-1}. \tag{3.7}$$

We now look for a finite sum $B_i = \sum_i B_{1,i} \in I_{-1}$ such that

$$A^*A + (B_0 + B_1)^*(B_0 + B_1) - c^2 I \in I_{-2}. \qquad (3.8)$$

We need $\sum_i B_0^* B_{1,i} + B_{1,i}^* B_0 + \sum_{i,j} B_{1,i}^* B_{1,j} = \tilde{B}_1 \mod I_{-2}$; since each $B_{1,i}^* B_{1,j} \in I_{-2}$ and the principal symbol of the $\tilde{B}_{1,i}$ are real, by (1.39) we may take $b_{1,i} = \frac{1}{2}\sigma_0(B_0)^{-1}\sigma_0(\tilde{B}_{1,i}) \in S^{p_i,l_i}(\mathbb{R}^n \times \mathbb{R}^n; \Delta, \Sigma)$ with $p_i + l_i = -1$, $p_i \leqslant -k/2$. To each $b_{1,i}$ there corresponds an operator $B_{1,i} \in I^{p_i,l_i}(\Delta, \Lambda)$ and, letting $B_1 = \sum_i B_{1,i}$, (3.8) is satisfied. Continuing in this fashion, we may find operators $B_i \in I_{-i}$ such that

$$A^*A + \left(\sum_{i=1}^{j} B_i\right)^* \left(\sum_{i=1}^{j} B_i\right) - c^2 I \in I_{-(j+1)}, \qquad \forall_j. \qquad (3.9)$$

Asymptotically summing, there is a $B \in I_0$ such that $A^*A + B^*B - c^2 I \in \bigcap_{j=1}^{\infty} I_{-j} = I^{-k/2}(\Lambda)$, yielding the L^2 boundedness of A.

On the other hand, if (p, l) lies on the other edge $p = -k/2$, $p + l < 0$, then we simply take $B_0 = I$, $c = 1$, and $A^*A + B_0^* B_0 - I \in I^{2p+k/2, 2l-k/2}(\Delta, \Lambda) \subset I_{2(p+l)}$. We now proceed to find operators $B_j \in I_{2j(p+l)}$, $j = 1, 2, \ldots$, as above, so that (3.9) is satisfied; the remainder of the proof is the same.

$$\text{Q.E.D.}$$

The preceding theorem can be applied immediately to obtain estimates for microlocal powers of a real principal type operator. Let $P(x, D)$ be a properly supported mth order pseudodifferential operator of real principal type, i.e., $p_m(x, \xi)$ real and $\nabla p_m(x, \xi) \neq 0$ on $\Sigma = \{(x, \xi): p_m(x, \xi) = 0\}$. For $\lambda \in \mathbb{C}$, define P^λ as in [I] or [A-U]. In [A-U] it is shown that $P^\lambda \in I^{(m-1)\text{Re}(\lambda)-1/2, \text{Re}(\lambda)+1/2}$, where Δ is the flowout of Σ. By Theorem 3.3, with $k = 1$, P^λ will be smoothing of order s_0 if $\max(m \text{Re}(\lambda), (m-1) \text{Re}(\lambda)) \leqslant -s_0$. Thus, we have

THEOREM 3.10. (a) $P^\lambda: H^s_{\text{comp}} \to H^{s-m\text{Re}(\lambda)}_{\text{loc}}$, $\text{Re}(\lambda) \geqslant 0$.

(b) $P^\lambda: H^s_{\text{comp}} \to H^{s-(m-1)\text{Re}(\lambda)}_{\text{loc}}$, $\text{Re}(\lambda) \leqslant 0$.

To give a more substantial application of (3.3), for $k = 1$, we now formulate some diffeomorphism-invariant classes of singular integral operators with conical singularities. Since all of the results are local, we will continue to work in $\mathbb{R}^n$. Fix an integer m, $2 \leqslant m \leqslant n-1$. Let $\gamma^0: \mathbb{R}^n \times S^{m-1} \times \mathbb{R} \to S^{n-1}$ be a C^∞ map such that for each $x \in \mathbb{R}^n$, the map $S^{m-1} \ni \omega \to \gamma(x, \omega, 0) \in S^{n-1}$ is an embedding with image $\gamma_x \subset S^{n-1}$. Define $\gamma: \mathbb{R}^n \times S^{m-1} \times \mathbb{R} \to \mathbb{R}^n$ by

$$\gamma(x, \omega, r) = x + r\gamma^0(x, \omega, r), \qquad (3.11)$$

and let $\Gamma_x = \{\gamma(x, \omega, r): (\omega, r) \in S^{m-1} \times \mathbb{R}\}$; then Γ_x is a cone-like variety in $\mathbb{R}^n$ with vertex at x, and the Γ_x's form a smooth family of such. Set

$$\Gamma = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n: y \in \Gamma_x\}. \tag{3.12}$$

Near $\Delta_{\mathbb{R}^n}$, Γ is smooth and of dimension $n + m$.

Next consider a smooth family of pseudodifferential kernels supported on the Γ_x's. Let $K \in \mathcal{D}'(\mathbb{R}^n \times S^{m-1} \times \mathbb{R})$ be a distribution which is smooth in x and ω, i.e., $WF(K) \subset \{(x, \omega, r, \xi, \Omega, \rho) \in T^*(\mathbb{R}^n \times S^{m-1} \times \mathbb{R}): \xi = 0, \Omega = 0\}$. Then by a standard result on restrictions of distributions, each $K(x, \omega, \cdot) \in \mathcal{D}'(\mathbb{R})$ is well defined. We will further assume that K has the specific form

$$K(x, \omega, r) = \int_{\mathbb{R}} e^{ir\rho} a(x, \omega, \rho)\, d\rho, \tag{3.13}$$

where $a \in S_{1,0}^\mu(\mathbb{R}^n \times S^{m-1} \times \mathbb{R} \times (\mathbb{R}\setminus 0))$. We have $WF(K) \subset \{(x, \omega, r, \xi, \Omega, \rho): \xi = 0, \Omega = 0, r = 0\}$. Define a distribution $\mathcal{K} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\langle \mathcal{K}, f \rangle = \int_{\mathbb{R}^n \times S^{m-1} \times \mathbb{R}} K(x, \omega, r)\, f(x, \gamma(x, \omega, r))\, dx\, d\omega\, dr,$$

$$f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n); \quad (3.14)$$

and define an operator with Schwartz kernel $\mathcal{K}$

$$Tf(x) = \int_{S^{m-1} \times \mathbb{R}} K(x, \omega, r)\, f(\gamma(x, \omega, r))\, d\omega\, dr, \qquad f \in C_0^\infty(\mathbb{R}^n). \tag{3.15}$$

We wish to find conditions on Γ so that $T \in I(\Delta, \Lambda)$ as in Theorem 3.3; to do so, we use the characterizations of $I(\Delta, \Lambda)$ discussed in Section 1.

First, let

$$N^*\Gamma' = \{(x, \xi, y, \eta) \in T^*(\mathbb{R}^n \times \mathbb{R}^n)\setminus 0: (x, y) \in \Gamma\setminus\Delta, (\xi, -\eta) \perp T_{(x, y)}\Gamma\}. \tag{3.16}$$

PROPOSITION 3.17. $WF(\mathcal{K})' \subset \Delta_{T^*\mathbb{R}^n} \cup N^*\Gamma'$ and thus

$$WF(Tf) \subset WF(f) \cup (N^*\Gamma' \circ WF(f)).$$

Proof. Let $g: \mathbb{R}^n \times S^{m-1} \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$, $g(x, \omega, r) = (x, \gamma(x, \omega, r))$. Then $\mathcal{K}$ is the pushforward of K by g, $\mathcal{K} = g_* K$, in the sense of [H I]. Thus, $WF(\mathcal{K}) \subset \{(x, \xi, y, \eta): dg^*(x, \omega, r)(\xi, \eta) \in WF(K)$, some $(\omega, r) \in S^{m-1} \times \mathbb{R}\}$. But $dg(x, \omega, r)(X, \Omega, R) = (X, X + R\gamma^0(x, \omega, r) + 0(r)) \Rightarrow dg^*(x, \omega, r)(\xi, \eta) = (\xi + \eta + 0(r), 0(r), (\eta \cdot \gamma^0))$. Hence, $WF(\mathcal{K}) \subset$

$\{(x, \xi, \nu, \eta): x = y, \xi + \eta = 0\} \cup \{(x, \xi, y, \eta): (x, y) \in \Gamma \backslash \Delta,$
$(\xi, \eta) \perp T_{(x, y)} \Gamma\} \subset \Delta'_{T^* \mathbb{R}^n} \cup N^* \Gamma.$ Q.E.D.

This s suggestive of (1.8), but in order to go further we need that $N^* \Gamma'$ be contained in a smooth Lagrangian that intersects $\Delta_{T^* \mathbb{R}^n}$ cleanly and in fact is contained in the flowout of a smooth, codimension 1 involutive submanifold $\Sigma \subset T^* \mathbb{R}^n \backslash 0$. Thus, we at least need to assume

$$N^* \Gamma' \text{ extends smoothly past } (T^*(\mathbb{R}^n \times \mathbb{R}^n) \backslash 0)|_{\Delta_{\mathbb{R}^n}}, \qquad (3.18)$$

i.e., $\overline{N^* \Gamma'}$ is smooth. We will denote $\overline{N^* \Gamma'}$ by Λ.

To better understand what condition (3.18) means in terms of the cones Γ_x, consider the intersection $\Delta_{T^* \mathbb{R}^n} \cap \overline{N^* \Gamma'}$. Without assuming (3.18), this is of the form Δ_Σ for some closed conic set $\Sigma \subset T^* \mathbb{R}^n \backslash 0$. By the proof of (3.17), $\lim_{r \to 0} N^* \Gamma' \cap T^*_{(x, \gamma(x, \omega, r))}(\mathbb{R}^n \times \mathbb{R}^n) = \{(x, \xi, x, \xi): \xi \perp \text{span} \{\gamma^0(x, \omega, 0), d_\omega \gamma^0(x, \omega, 0)(T_\omega S^{m-1})\}\}$, so that

$$\Sigma \cap (T^*_x \mathbb{R}^n \backslash 0) = \bigcup_{\sigma \in \gamma_s} (\text{span} \{\sigma, T_\sigma \gamma_x\})^\perp, \qquad (3.19)$$

which is an $(m-1)$-parameter union of $(n-m)$-planes in $T^*_x \mathbb{R}^n$. Thus, if Σ is smooth of codimension k, we must have $1 \leqslant k \leqslant m$.

Let us consider the case $k = 1$, which is the one of interest, in more detail; it will be seen that (3.18) is actually a curvature condition. In light of (3.19) it is natural to ask: Which $(m-1)$-dimensional submanifolds $S \subset S^{n-1}$ have the property that $\Omega = \bigcup_{\sigma \in S} (\text{span} \{\sigma, T_\sigma S\})^\perp$ is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^{n^*}$? Working locally on S near a point σ_1, let $\{e_1, ..., e_{m-1}\}$ and $\{\omega^m, ..., \omega^{n-1}\}$ be orthonormal frames for TS and $N^* S$, respectively, with respect to the standard metric on S^{n-1}. Then we may locally parameterize Ω by $S \times (\mathbb{R}^{n-m} \backslash 0) \ni (\sigma, \theta) \to \sum_{j=m}^{n-1} \theta_j \omega^j(\sigma)$, where we are making the natural identification of the ω^j's as elements of $\sigma^\perp \subset \mathbb{R}^{n^*}$. This will be an immersion at (σ, θ) iff $\{\nabla_{e_i} (\sum_j \theta_j \omega^j(\sigma))\}_{j=1}^{n-m}$ are linearly independent modulo $\text{span} \{\omega^n(\sigma), ..., \omega^{n-1}(\sigma)\} = N^*_\sigma S$; by the Gauss equation [K, p. 90], this will hold iff the second fundamental form of S in S^{n-1} is nonsingular in the direction $\sum_j \theta_j \omega^j$. Since we want this to hold for all $\theta \in \mathbb{R}^{n-m} \backslash 0$, we are naturally led to the following condition on S: denoting the second fundamental form of S at σ in the normal direction ν by A^ν,

$$A^\nu: T_\sigma S \to T_\sigma S \text{ is nonsingular,} \qquad \forall \nu \in N^*_\sigma S \backslash 0, \sigma \in S. \qquad (3.20)$$

The above discussion shows that Ω is immersed if (3.20) holds, with Ω being embedded if we impose the additional global assumption

$$\Omega \text{ has no self-crossings.} \qquad (3.21)$$

Applying this to $S = \gamma_x$ as above, one sees that if each γ_x satisfies (3.20) and (3.21), then Σ, given by (3.19), is smooth and codimension 1 in $T^* \mathbb{R}^n \backslash 0$. Furthermore, it is straightforward to see that $\Lambda = \overline{N^* \Gamma'}$ is smooth and intersects $\Delta_{T^* \mathbb{R}^n}$ cleanly at Δ_Σ.

Now, with (3.15) in mind, define $S: C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n \times S^{m-1} \times \mathbb{R})$ by $Sf(x, \omega, r) = f(\gamma(x, \omega, r))$. The Schwartz kernel of S is δ_W, where $W = \{(x, \omega, r, y): y = x + r\gamma^0(x, \omega, r)\}$, which is smooth and codimension n in $\mathbb{R}^n \times S^{m-1} \times \mathbb{R} \times \mathbb{R}^n$, and thus $S \in I^{-m/4}(C_S)$, $C_S = N^* W' \subset (T^*(\mathbb{R}^n \times S^{m-1} \times \mathbb{R}) \backslash 0) \times (T^* \mathbb{R}^n \backslash 0)$. Then we find the oscillatory integral representation for the Schwartz kernel of S

$$K_S(x, \omega, r, y) = \int e^{i((x-y) \cdot \xi + r\gamma_0(x,\omega,r) \cdot \xi)} b(x, y, \omega, r, \xi) \, d\xi \qquad (3.21)$$

with $b \in S^0(\mathbb{R}^n \times \mathbb{R}^n \times S^{m-1} \times \mathbb{R} \times (\mathbb{R}^n \backslash 0))$. Then we have

$$\mathscr{K}(x, y) = \int e^{i(r\sigma + (x-y) \cdot \xi + r\gamma_0(x,\omega,r) \cdot \xi)} a(x, \omega, r, \sigma) \, b(x, y, \omega, r, \xi) \, d\xi \, d\omega \, dr.$$

$$(3.22)$$

We now use the characterization (1.35) of elements in $I(\Delta, \Lambda)$, to prove

THEOREM 3.23. *Let T be as in (3.15) with the family of cones satisfying (3.18). Then $T \in I^{-m/2, \mu + m/2}(\Delta, \Lambda)$ with Δ the diagonal and $\Lambda = \overline{N^* \Gamma'}$ as in (3.16).*

Using Theorem 3.23 and Theorem 3.3 we obtain

THEOREM 3.24. *Let T be as in Theorem 3.23. Then*

$$T: H_{\text{comp}}^s(\mathbb{R}^n) \to H_{\text{loc}}^{s+s_0}(\mathbb{R}^n), \qquad \forall s \in \mathbb{R},$$

if

$$\max\left(-\frac{m-1}{2}, \mu\right) \leq -s_0.$$

Proof of Theorem 3.23. Let P_i, $1 \leq i \leq r$, be pseudodifferential operators of order 1 with principal symbol p_i vanishing on Δ' and Λ'. Then

$$P_i \mathscr{K} = \int e^{i\phi(x, y, \omega, r, \xi, \sigma)} p_i(x, y, \xi, r\nabla_x \gamma_0 \cdot \xi) \, ab \, d\xi \, d\sigma \, d\omega \, dr + B_i, \qquad (3.25)$$

where

$$\phi(x, y, \omega, r, \xi, \sigma) = r\sigma + (x-y) \cdot \xi + r\gamma_0(x, \omega, r) \cdot \xi \qquad (3.26)$$

and $B_i \in \mathscr{A}_{\mu,0}$ with

$$\mathscr{A}_{\alpha,\beta} = \{\mu \in D'(\mathbb{R}^n \times \mathbb{R}^n); \ \mu \text{ is of the form (3.22) with } a \in S^\alpha, b \in S^\beta\}.$$

$$(3.27)$$

Using that p_i vanishes on Δ' and the fact that $(x_i - y_i)e^{i\phi} = D_{\xi_i}e^{i\phi}$, $re^{i\phi} = D_\sigma e^{i\phi}$ we obtain by integrating by parts in the ξ_i and σ variables

$$P_i \mathscr{K} \in \mathscr{A}_{\mu-1,1} + \mathscr{A}_{\mu,0}.$$

$$(3.28)$$

Repeating this argument inductively we get

$$P_1 \cdots P_r K \in \mathscr{A}_{\mu-r,r} + \mathscr{A}_{\mu-r-1,r-1} + \cdots + \mathscr{A}_{\mu,0}.$$

$$(3.29)$$

Now let us consider the terms $\mathscr{A}_{\mu-j,j}$ with $\mu - j \leqslant -2$. For elements in this class we can intergrate out the σ-variable and we obtain for any $A \in \mathscr{A}_{\mu-i,j}$, $j \geqslant \mu + 2$,

$$A = \int e^{i(x-y)\cdot\xi + r\gamma_0(x,\omega,r)\cdot\xi} c(r, x, \omega, \xi) \, d\xi \, d\omega \, dr$$

$$(3.30)$$

with $c \in C^{j-\mu-2}(\mathbb{R}, S^j(\mathbb{R} \times S^{m-1} \times (\mathbb{R}^n \backslash 0)))$. We note that Λ' is parameterized by the phase function $\chi(x, y; r/|\xi|, \omega/|\xi|, \xi)$ with

$$\chi(x, y, r, \omega, \xi) = (x-y)\cdot\xi + r\gamma_0(x, \omega, r)\cdot\xi.$$

Now, we use the fact that the principal symbol vanishes on Λ' to conclude that $P_i K$ is of the form (3.30) for some

$$c \in C^{j-\mu-2}(\mathbb{R}, S^j(\mathbb{R} \times S^{n-1} \times \mathbb{R}^n \backslash 0)).$$

Summarizing the arguments above we have proved that

$$P_1 \cdots P_r K \in \mathscr{A}_{-2,\mu+2} + \mathscr{A}_{-1,\mu+1} + \cdots + \mathscr{A}_{\mu,0}.$$

$$(3.31)$$

Now to define the oscillatory integrals in $A_{\alpha,\beta}$ with $|\alpha| + |\beta|$ bounded we have to integrate by parts at most a finite number of derivatives in the x, y variables, proving that $\mathscr{A}_{-2} + \mathscr{A}_{-1,\mu+1} + \cdots + \mathscr{A}_{\mu,0}$ is contained in a fixed Sobolev space. This concludes the proof that $T \in I(\Delta, \Lambda)$. To determine the order we simply observe that T is a pseudodifferential operator of order μ on the diagonal and a Fourier integral operator of order $-m/2$ on Λ away from the intersection.   Q.E.D.

4. Estimates for Restricted X-Ray Transforms

Operators with conical singularities of the type considered in Section 3 arise naturally in integral geometry. Combining the results of Section 3

with those of [G-U], we obtain sharp local L^2 estimates for the restriction
of the X-ray transform to an admissible line (or, more generally, geodesic)
complex, $\mathscr{C}$. (Wang [W] obtained global L^p estimates for some line
complexes in $\mathbb{R}^n$.) For a large number of general (inadmissible) complexes,
it will be shown that one actually obtains better estimates; this reflects the
fact that the Gelfand condition for admissibility forces the conormal bundle
of the point-line (or point-geodesic) relation to sit in $T^*M \times T^*\mathscr{C}$ in a
more singular fashion that it does in general.

To put this in context, let X and Y manifolds of dimension n, $C \subset$
$(T^*X\backslash 0) \times (T^*Y\backslash 0)$ a canonical relation, and π and ρ the restrictions to
C of the projections from $(T^*X\backslash 0) \times (T^*Y\backslash 0)$ onto $T^*Y\backslash 0$ and $T^*X\backslash 0$,
respectively. Let $R \in I^m(X, Y; C)$ be a Fourier integral operator. At a point
$c_0 \in C$, the differential $d\pi(c_0)$ is invertible (i.e., has rank $2n$) iff $d\rho(c_0)$ is, in
which case c_0 has a neighborhood in C which is the graph of a canonical
transformation from $T^*Y\backslash 0$ to $T^*X\backslash 0$. If $d\pi$ and $d\rho$ have rank $2n$
everywhere, then C is a local canonical graph and $R: H^s_{\text{comp}}(Y) \to H^{s-m}_{\text{loc}}(X)$,
$\forall s \in \mathbb{R}$ [H I]. On the other hand, by another result of Hörmander, if $d\pi$
and $d\rho$ drop rank by no more than 1 at each point of C, $R: H^s_{\text{comp}}(Y) \to$
$H^{s-m-1/2}_{\text{loc}}(X)$ (under the mild additional assumption that $\tilde{\pi}: C \to Y$ and
$\tilde{\rho}: C \to X$ are submersions.) In general, this result is sharp, as the case when
C is the flowout of a codimension 1 conic submanifold in $T^*X\backslash 0$ shows.
One expects, of course, that for a C which is not a local canonical graph
but for which π and ρ become singular in specific ways, there should be a
loss of s_0 derivatives, for some sharp value of s_0, $0 < s_0 < \frac{1}{2}$. An example is
provided by the work of Melrose and Taylor [M-T] in scattering theory.
There, C's for which π and ρ have at most a Whitney fold were introduced
and termed folding canonical relations. It was shown that, using canonical
transformation of $T^*X\backslash 0$ and $T^*Y\backslash 0$, any such C can be conjugated to a
single normal form; on the operator level, any $R \in I^m(X, Y; C)$ can be
conjugated by elliptic Fourier integral operators to an Airy operator on $\mathbb{R}^n$
with symbol in $S^{m+1/6}_{1,0}$, giving the estimate $R: H^s_{\text{comp}}(Y) \to H^{s-m-1/6}_{\text{loc}}(X)$.
We now use the results of Section 3 to show that for a certain class of
canonical relations arising in integral geometry, somewhat more singular
than the folding canonical relations, there is a loss of $\frac{1}{4}$ derivative. It should
be noted that there is no single normal form for the canonical relations
described below; there are already obstructions to equivalence at the
quartic terms in a formal power series calculation.

DEFINITION 4.1 [G-U]. A *fibered folding canonical relation* $C \subset$
$(T^*X\backslash 0) \times (T^*Y\backslash 0)$ is a canonical relation such that

  (4.1a) at each point of C, π is either a local diffeomorphism or a
Whitney fold; letting $L \subset C$ denote the fold hypersurface;

(4.1ɔ) at each point $l_0 \in L$, $d\rho$ drops rank simply by one and $\ker(d\rho(l_\iota)) \subset T_{l_0}L$, so that $\rho|_L$ has one-dimensional fibers, and the map $d\bar{\rho}$: $\rho^{-1}(\rho(l_0)) \to G_{2n-1,2n}(T_{\rho(l_0)}T^*X)$ defined by $d\bar{\rho}(l) = d\rho(l)(T_l C)$ is an immersicn;

(4.1ɔ) $\pi(L) \subset T^*Y \setminus 0$ is embedded, $\rho|_{C \setminus L}$ is $1-1$, and $\rho(L) \subset T^*X \setminus 0$ is embedded symplectic; and

(4.1d) the fibers of ρ are the lifts by π of the Hamiltonian bicharacteristics of $\pi(L)$.

(This ɟype of canonical relation has also been considered by Guillemin [Gu II]. who pointed out that (4.1d) is in fact redundant.)

As is shown in [G-U], canonical relations of this type arise naturally in integral ɟeometry. If $\mathcal{M}$ is the $(2n-2)$-dimensional manifold of geodesics on a riɛmanniom manifold (M, g) (when this makes sense) satisfying certain assumptions and $\mathscr{C}^n \subset \mathcal{M}$ is a globally admissible geodesic complex satisfyinɡ a curvature condition, ((2.23) in [G-U]), then the restricted X-ray trɑnsform (integration over geodesics) $\mathscr{R}_{\mathscr{C}} f = (\mathscr{R}_{1,n} f)|_{\mathscr{C}}$ is injective modulo C^∞ on distributions of compact support with appropriate restrictions on their wave front sets. An important ingredient in the proof is the fact that $\mathscr{R}_{\mathscr{C}} \in I^{-1/2}(\mathscr{C}, M; C)$ with C a fibered folding canonical relation as above. Iɔ addition, $C' \circ C \subset \Delta_{T^*M} \cup \Lambda_{\pi(L)} \subset (T^*M \setminus 0) \times (T^*M \setminus 0)$, where Δ_{T^*M} is the diagonal and $\Lambda_{\pi(L)}$ is the flowout of $\pi(L)$. The symbol of $\mathscr{R}_{\mathscr{C}}^t \circ \mathscr{R}_{\mathscr{C}}$ is then computed, using only the fact that $\pi^*(\omega_{T^*M})$ is a folded symplectic form on C (for definitions and related material, see [G-U, Gu II]), and is shown to be elliptic, allowing the construction of a relative left-paraɾnetrix for $\mathscr{R}_{\mathscr{C}}$.

THEORɛM 4.2. *Let (M, g) be an n-dimensional riemannian manifold, and $\mathscr{C}$ be an adɹ1issible geodesic complex satisfying the assumptions of Theorem 2.7 of [G-U], and with the projection $\{(x, \gamma) \in M \times \mathscr{C} : x \in \gamma\} \to M$ proper. Then for any ɔlosed set $K \subset T^*M \setminus 0$ contained in the support of the Crofton symbol aɪnd with compact projection in M disjoint from the critical set of $\mathscr{C}$, there is a Fourier integral operator $B \in I^{-1/2}(\Lambda_{\pi(L)})$ such that*

$$\|\mathscr{R}_{\mathscr{C}} f\|_{H^{s+1/4}} \leqslant C_{s,K} \|f\|_{H^s}, \quad f \in H^s(M) \cap \mathscr{E}'_K, s \geqslant -\frac{1}{4}, \quad (4.3)$$

and, for ɪ $= -\frac{1}{4}$,

$$C'_k \|(J - B)f\|_{H^{-3/4}} \leqslant \|\mathscr{R}_{\mathscr{C}} f\|_{L^2} \leqslant C_k \|f\|_{H^{-1/4}}, \quad f \in H^{-1/4}(M) \cap \mathscr{E}'_K.$$

$$(4.4)$$

Proof. The operator $T = \mathscr{R}_{\mathscr{C}}^t \circ \mathscr{R}_{\mathscr{C}}$ can be described explicitly as follows. For $x \in M$, let $\mathscr{C}_x = \{\gamma \in \mathscr{C} : x \in \gamma\}$ be the geodesics of the complex that pass through ɪx; by assumption, we are working near a generic point x, so that

$\mathscr{C}_x$ is one-dimensional. Assume for the moment that $\mathscr{C}_x$ is connected and hence diffeomorphic to S^1. Parameterizing the geodesics in $\mathscr{C}_x$ by an arc-length parameter vanishing at x, we obtain locally smooth maps γ^0: $M \times S^1 \times \mathbb{R} \to S^{n-1}$ and $\gamma: M \times S^1 \times \mathbb{R} \to M$ as in Section 3. For some smooth, nonvanishing density $d\mu_x(x, \omega, r)$ on $S^1 \times \mathbb{R}$,

$$Tf(x) = \int_{S^1 \times \mathbb{R}} f(\gamma(x, \omega, r)) \, d\mu_x(\omega, r), \qquad (4.5)$$

which is of the form (3.15) with $m = 1$ and $a \in S^{-1}_{1,0}((M \times S^1) \times (\mathbb{R}\backslash 0))$. In general, T is a sum of such operators. Thus, by the discussion in Section 3, $T \in I^{-1,0}(\varDelta, \varLambda_{\pi(L)})$, the smoothness of $\varLambda = \varLambda_{\pi(L)}$ following from the curvature assumption on $\mathscr{C}$ alluded to above. By Theorem 3.3, $T: H^s_{\text{comp}}(M) \to H^{s+1/2}_{\text{loc}}(M)$, $\forall s \in \mathbb{R}$. By duality $\mathscr{R}_{\mathscr{C}}: H^{-1/4}_{\text{comp}}(M) \to L^2(\mathscr{C})$.

Now recall that if $\gamma \in \mathscr{M}$, the tangent space $T_\gamma \mathscr{M}$ can be identified with $J^\perp_\gamma$, the space of Jacobi vector fields along γ which are orthogonal to γ. Denoting this isomorphism by $T_\gamma \mathscr{M} \ni X \leftrightarrow \tilde{X}(s) \in J^\perp_\gamma$, we also have an identification between smooth vector fields on $\mathscr{M}$ (or $\mathscr{C}$) and smooth families of Jacobi fields, which we denote by $X(\gamma) \leftrightarrow \tilde{X}(\gamma, s)$. Furthermore, we have

$$X(\mathscr{R}_{\mathscr{C}} f)(\gamma) = \mathscr{R}_{\mathscr{C}}(\tilde{X}(\gamma, \cdot) f)(\gamma), \qquad f \in C^\infty_0(M) \qquad (4.6)$$

if X is tangent to $\mathscr{C}$. Hence, if P is an mth order partial differential operator on $\mathscr{C}$, there is a smooth family of mth order operators $\tilde{P}(\gamma, s)$ along the geodesics of $\mathscr{C}$ such that

$$P(\mathscr{R}_{\mathscr{C}} f)(\gamma) = \mathscr{R}_{\mathscr{C}}(\tilde{P}(\gamma, \cdot) f)(\gamma), \qquad f \in C^\infty_0(M). \qquad (4.7)$$

For such a P, $P\mathscr{R}_{\mathscr{C}}: H^{m-1/4}_{\text{comp}}(M) \to L^2(\mathscr{C})$ and thus, since P is arbitrary, we have $\mathscr{R}_{\mathscr{C}}: H^{m-1/4}_{\text{comp}}(M) \to H^m(\mathscr{C})$, $m = 0, 1, 2, \dots$. By interpolation we obtain (4.3) for $s \in \mathbb{R}$, $s \geqslant -\frac{1}{4}$.

It is shown in [G-U] that T is elliptic on the support of the Crofton symbol and there exists a $B \in I^{-1/2}(\varLambda_{\pi(L)})$ and a $T^{-1} \in I^{0,1}(\varDelta, \varLambda_{\pi(L)})$ such that $T^{-1} \circ T = I - B$ on $\mathscr{E}'_K$; by Theorem 3.3, $T^{-1}: H^s_{\text{comp}}(M) \to H^{s-1}_{\text{loc}}(M)$, $\forall s \in \mathbb{R}$. From this, since (4.3) implies that $\mathscr{R}^t_{\mathscr{C}}: H^s_{\text{comp}} \to H^{s+1/4}_{\text{loc}}$ for $s \leqslant 0$, it follows that the relative left-parametrix $T^{-1} \circ \mathscr{R}^t_{\mathscr{C}}$ for $\mathscr{R}_{\mathscr{C}}$ maps $L^2_{\text{comp}}(\mathscr{C}) \to H^{-3/4}_{\text{loc}}(M)$, yielding (4.4). Q.E.D.

Remarks. (1) One expects (4.3) to hold for all $s \in \mathbb{R}$.[2]

(2) The discrepancy between the norms on the left- and right-hand sides of (4.4) arises for the same reason as the loss of 1 derivative in the parametrix of an operator of real principal type. This is already reflected in (3.10).

[2] The authors recently proved this. (See "Composition of Some Singular Fourier Integral Operators and Estimates for Restricted X-ray Transforms," preprint.)

We next show that the gain of $\frac{1}{4}$ derivative in (4.3) is in fact sharp. Let $M = \mathbb{R}^3$ with the Euclidean metric, and $\mathscr{C} \subset \mathscr{M} = M_{1,3}$ be the line complex consisting of all lines (light rays) on the light cone $\{x_3^2 = x_1^2 + x_2^2\}$ and their translates, i.e., $\mathscr{C}$ is all light rays in $2 + 1$-dimensional Minkowski space. Parametrize $\mathscr{C}$ by $\mathbb{R}^2 \times S^1$, with

$$\mathscr{R}_{\mathscr{C}} f\ x, y, \theta) = \int_{\mathbb{R}} f(x + t \cos \theta, y + t \sin \theta, t)\, dt, \qquad f \in \mathscr{E}'(\mathbb{R}^3). \tag{4.8}$$

A simple calculation yields

$$\mathscr{R}_{\mathscr{C}}^t\, g(x_1, x_2, x_3) = \int_0^{2\pi} g(x_1 - x_3 \cos(\theta), x_2 - x_3 \sin(\theta), \theta)\, d\theta, \qquad g \in \mathscr{E}'(\mathscr{C}). \tag{4.9}$$

Let $f = H(x_1 - x_3)\, \varphi(x_1, x_2, x_3)$, where H is the Heaviside function and $\varphi \in C_0^\infty$, $\varphi(0) \neq 0$. (We are trying to concentrate the Fourier transform of f along a line on the dual light cone.) Then $f \in H_{\mathrm{comp}}^{1/2 - \varepsilon}(\mathbb{R}^3)$, $\forall \varepsilon > 0$. For $g \in C_0^\infty(\mathscr{C})$,

$$\langle \mathscr{R}_{\mathscr{C}} f, g \rangle = \langle f, \mathscr{R}_{\mathscr{C}}^t g \rangle$$

$$= \int g(x, y, \theta) \left[\int_{(1 - \cos(\theta))x_1 \leqslant x} \varphi(x + x_3 \cos(\theta), y \right.$$

$$\left. + x_3 \sin(\theta), x_3)\, dx_3 \right] dx\, dy\, d\theta. \tag{4.10}$$

Localizing near $x, y, \theta \sim 0$, we see that near the origin $\mathscr{R}_{\mathscr{C}} f$ is smooth in y and essentially homogeneous of degree 0 in (x, θ) with respect to the dilations $(x, \theta) \to (\delta^2 x, \delta\theta)$; thus the partial Fourier transform is roughly homogeneous of degree -3 with respect to these dilations. From this a routine calculation shows that $\mathscr{R}_{\mathscr{C}} f \in H^{3/4 - \varepsilon}(\mathscr{C})$, $\forall \varepsilon > 0$, but $\mathscr{R}_{\mathscr{C}} f \notin H^{3/4}(\mathscr{C})$.

Now let us turn to general geodesic complexes $\mathscr{C}$ which do not satisfy Gelfand's admissibility criterion. For example, one may consider small deformations, in the C^∞ topology, of a fixed admissible complex, $\mathscr{C}_0$. Since the canonical relation $C_0 = N^* Z_0'$, where $Z_0 \subset M \times \mathscr{C}_0$ is the point-geodesic relation $\{(x, \gamma): x \in \gamma\}$, for $\mathscr{C}_0$ is a fibered folding canonical relation, near a point $c_0 \in C_0$ there are local (nonhomogeneous) coordinates (x, ξ), vanishing at c_0, and local (noncanonical) coordinates on $T^* M$ and $T^* \mathscr{C}_0$, such that

$$\pi(x, \xi) = \left(x, \xi', \frac{\xi_n^2}{2} \right) \qquad \text{(Whitney fold)}$$

$$\rho(x, \xi) = (x', x_n \xi_n, \xi) \qquad \text{(fibered fold or blow-down)}. \tag{4.11}$$

Perturbing $\mathscr{C}_0$ into a general nearby complex $\mathscr{C}$ corresponds to perturbing the maps π and ρ in C^∞. Since Whitney folds are stable, a small perturbation of π will still be a Whitney fold. On the other hand, a *generic* small perturbation of ρ will be either a Whitney fold ($S_{1,0}$ singularity), a simple cusp ($S_{1,1,0}$ singularity), or higher singularity (Golubitsky and Guillemin [Go-Gu]). Suppose that the former holds; then the corresponding geodesic complex $\mathscr{C}$ has a restricted X-ray transform $\mathscr{R}_{\mathscr{C}} \in I^{-1/2}(\mathscr{C})$, with C belonging to the class of folding canonical relations, introduced by Melrose and Taylor [M-T], near c_0. It follows from the results of [M-T] that, microlocally near c_0, $\mathscr{R}_{\mathscr{C}}$ can be conjugated by elliptic zeroth order Fourier integral operators to an Airy operator on $\mathbb{R}^n$ proving

THEOREM 4.12. *Let $\mathscr{C} \subset \mathscr{M}$ be a geodesic complex and $z = \{(x, \gamma) \in M \times \mathscr{C} : x \in \gamma\}$. If $C = N^*Z' \subset (T^*\mathscr{C} \backslash 0) \times (T^*M \times 0)$ is a folding canonical relation, then $\mathscr{R}_{\mathscr{C}} : H^s_{\mathrm{comp}}(M) \to H^{s+1/3}_{\mathrm{loc}}(\mathscr{C})$, $\forall s \in \mathbb{R}$.*

Explicit examples of families of geodesic complexes $\{\mathscr{C}_\varepsilon\}_{\varepsilon \in \mathbb{R}}$, with $\mathscr{C}_0$ admissible but $\mathscr{C}_\varepsilon$ satisfying the hypotheses of (4.12) for $\varepsilon \neq 0$ are easily constructed. For example, the line complex associated with the light cone in $\mathbb{R}^3$ considered above can be perturbed as follows. Equip $\mathbb{R}^3$ with the Heisenberg group structure with Planck's constant $\varepsilon : (x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \varepsilon(x_1 y_2 - x_2 y_1))$, and let $\mathscr{C}_\varepsilon$ be the line complex consisting of all light rays through $(0, 0, 0)$ and their left-translates. Then $\mathscr{C}_0$ is admissible and $\mathscr{R}_{\mathscr{C}_0} : H^s_{\mathrm{comp}} \to H^{s+1/4}_{\mathrm{loc}}$ by Theorem 4.2, but for $\varepsilon \neq 0$ small, $\mathscr{C}_\varepsilon$ has a canonical relation C_ε which is folding, and $\mathscr{R}_{\mathscr{C}_\varepsilon} : H^s_{\mathrm{comp}} \to H^{s+1/3}_{\mathrm{loc}}$ by (4.12).

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